Minimax Optimization of Continuous Search Efforts for the Detection of a Target

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Abstract: Analytical resolution of search theory problems, as formalized by B.O. Koopman, may be applied with some model extension to various resource management and data fusion issues. Such method is based on a probabilistic prior about the target. Even so, this approximation forbids any reactive behavior of the target. As a preliminary step towards reactive target study stands the problem of resource placement under a minimax game context. This report is related to Nakai's work about the game placement of resources for the detection of a stationary target. However, this initial problem is extended by adding new and more general constraints, allowing a more subtle modeling of the target and resource behaviors.

Key-words: Sensor & Resource Management, target detection, Search game, Search theory, Resource allocation

(Résumé : tsvp)

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Optimisation minimax d'un effort de recherche continu pour la detection d'une cible

Résumé : La résolution analytique des problèmes de Search Theory, comme l'a formalisé B.O. Koopman, peut être appliquée avec quelques extensions de modèle à des situation variées de gestion de ressource et de fusion de données. Ce genre de méthode se base sur un a priori probabiliste de la cible. Une telle approximation ne permet pas de modéliser l'éventuelle réactivité de la cible. Une étape préliminaire à l'étude de cible réactive réside dans les problèmes de placement des ressources dans le contexte de jeux mini-max. Ce rapport prend comme point de départ le travail de Nakai concernant le jeu de l'allocation des ressources pour la détection d'une cible stationnaire. Ce problème initial a été étendu par l'adjonction de nouvelles contraintes, permettant des modélisations plus souple du comportement de la cible et des ressources.

Mots-clé : Gestion de capteurs, Détection, Jeux de recherche, Théorie de la recherche, Allocation de ressources

Notations

- $\varphi(x)$: Search effort,
- ϕ_o: Total amount of search effort,
- $\alpha(x)$: Probabilistic target distribution,
- A_o: Total target probability,
- $p_x(\varphi(x))$: Conditional non detection probability.

1 Introduction

The initial framework of Search Theory [4][2][3], introduced by B.O. Koopman and his colleagues, sets the general problem of the detection of a target in a space, in view of optimizating the detection resources. A thorough extension of the prior formalism has been made by Brown towards the detection at several periods of search [5][6]. These simple but meaningful formalism were also applied to various resource management and data fusion issues [7]. But, in all these problems, a probabilistic prior on the target was required. In addition, in case of moving target problems, a Markovian hypothesis is necessary for algorithmic reasons. While this formalism is sufficient for almost "passive" targets, it is useless when a target has a complex (and realistic) move. In a military context especially, the behavior of the "interesting" targets is not neutral and cannot be modeled by a simple probabilistic prior. A conceivable way for enhancing the prior on the target in a manner that involves more properly the complexity or the reactiveness of the target, is to consider a min-max game version of Koopman optimization problems. Nakai presented and solved it in [8] a game with placement of resources for the detection of a stationary target. In this work the constraints on game were given by the available placement of target and detection resources. Thus, constraints were defined at the *pure strategy* level. The purpose of this paper is to present an extension of Nakai's game by addition of new constraints defined on the set of available mixed strategies. In other words, constraints are now defined at the *mixed strategy* level. Before explaining properly the extended problem, we intend to give in this introduction a short description of Nakai's game.

Definitions: The searcher want to detect a target located in a search space E. To perform this detection, the searcher has available a total amount of (detection) resources ϕ_o. These resources may be put on each cell x of the search space E. Detection on cell x is a known function of the search effort put on x. For $x \in E$, the variable $\varphi(x)$ denotes the local amount of resources placed on cell x. A constraint naturally holds for the global amount of resources in use:

$$\int_E \varphi(x)\, dx \leq \phi_o \,.$$

Since detection is better when the whole resources are used, without restricting generality the previous constraint may be replaced by an equality one:

$$\int_E \varphi(x)\, dx = \phi_o \,. \tag{1}$$

The set of valid sharing functions φ is thus defined by:

$$\mathcal{R}(\phi) = \left\{ \varphi \in \mathbb{R}^{+E} \Big/ \int_E \varphi = \phi_o \right\} \,.$$

When local resource $\varphi(x)$ is put on cell x and target is located on x, the probability of non detection is given by value $p_x(\varphi(x))$, a conditional probability. This probability may depend upon x, since practically visibility and resource efficiency vary with the concerned cell. For x fixed, p_x decreases with the effort used and $p'_x < 0$. The detection follows the rule of decreasing return, so that p'_x increases strictly with φ. On the other hand, the target have the choice between available positions $\mathbb{T} \subset E$. Then, a game occurs between the searcher and the target. The searcher attempts to minimize the probability of non detection by optimizing the search resource sharing φ, while the target aim is to maximize the probability of non detection by choosing his position. The value of the game is given by $p_\Theta(\varphi(\Theta))$, for a target strategy Θ and a searcher strategy φ. This problem was solved by Nakai [8]. Since p is convex, it appears that the game is convex. Thus, there is a mixed optimal strategy for the target and a pure optimal strategy for the searcher. A mixed strategy for the target is given by a density probability α on the target position, with property $\alpha(E \setminus \mathbb{T}) = 0$. We denote:

$$\mathcal{P}(\mathbb{T}) = \left\{ \alpha \in {\rm I\!R^+}^E \Big/ \alpha(E \setminus \mathbb{T}) = 0 \text{ and } \int_E \alpha = 1 \right\} .$$

the set of such probabilities. For strategies (α, φ), the value of the game is then given by the averaged value (denoted P_{nd}) of the probability of non detection.

$$P_{nd}(\alpha, \varphi) = \int_E \alpha(x) p_x(\varphi(x))\, dx .$$

An optimal (min-max) couple of strategies (α_o, φ_o) is also defined by :

$$\left\{ \begin{array}{l} \alpha_o = \arg \max\limits_{\alpha \in \mathcal{P}(\mathbb{T})} \min\limits_{\varphi \in \mathcal{R}(\phi_o)} \displaystyle\int_E \alpha(x) p_x(\varphi(x))\, dx , \\ \\ \varphi_o = \arg \min\limits_{\varphi \in \mathcal{R}(\phi_o)} \max\limits_{\alpha \in \mathcal{P}(\mathbb{T})} \displaystyle\int_E \alpha(x) p_x(\varphi(x))\, dx . \end{array} \right.$$

Two optimality conditions are obtained, by differentiation around the optimal strategies:

$$\left\{ \begin{array}{l} \alpha_o(x)\, p'_x(\varphi_o(x)) = \eta \ , \text{ when } \alpha_o(x) > \dfrac{\eta}{p'_x(0)} \\ \\ \varphi_o(x) = 0 \ , \text{ else} \end{array} \right. \tag{2}$$

and

$$\exists \lambda \in {\rm I\!R}^+,\ \alpha_o(x) > 0 \Longrightarrow p_x(\varphi_o(x)) = \lambda . \tag{3}$$

We can recognize in (2) the classical optimality equation of de Guenin. By use of these equations, a mathematical solution of the problem is built. The first step is to verify the obviously intuitive result:

$$x \in \mathbb{T} \Longleftrightarrow \Big[\alpha_o(x) > 0 \text{ and } \varphi_o(x) > 0 \Big] . \tag{4}$$

Then, the combination of equations (1) and (3) yields $\int_{\mathbb{T}} p_x^{-1}(\lambda)\,dx = \phi_o$. Defining the function $\mathbb{P}$ by:

$$\mathbb{P}^{-1}(\lambda) = \int_{\mathbb{T}} p_x^{-1}(\lambda)\,dx\ ,$$

it follows that $\lambda = \mathbb{P}(\phi_o)$. At last, equations (2), (3) and (4) simplify and reduce to $\varphi_o(x) = p_x^{-1}(\mathbb{P}(\phi_o))$ and $\alpha_o(x) = \eta / p'_x(p_x^{-1}(\mathbb{P}(\phi_o)))$. Since α_o is a probability density, it follows that $\int_{\mathbb{T}} \alpha_o(x)\,dx = 1$. This property permits to find the dual variable η. After simplification, the simple formula $\eta = \mathbb{P}'(\phi_o)$ is obtained. Finally, the min-max optimal strategies (α_o, φ_o) are simply given by:

$$\boxed{\forall x \in \mathbb{T},\ \begin{cases} \alpha_o(x) = (p_x^{-1} \circ \mathbb{P})'(\phi_o) \\ \\ \varphi_o(x) = (p_x^{-1} \circ \mathbb{P})(\phi_o) \end{cases}} \tag{5}$$

The Nakai game problem also admits a mathematical solution. In fact, the game remodeling of the search problem yields some complexity simplification in comparison with the classical one-sided search problem [1]. However, there is no general mathematical solution. In the next section, an extension of Nakai problem will be considered. It is a min-max game, where constraints are given on the target mixed strategies. Such problem will be seen as a generalization of both Nakai game and de Guenin's problem, but is much more complex than these two parent problems. New properties will be established to handle these difficulties and an *original* algorithm will be presented.

2 Bounding constraints

In Nakai game, the prior on target is given by the set of available target positions. This hypothesis constitutes a prior more general and more flexible than a probabilistic density on target position, in particular for modeling uncertain targets. Nevertheless, it does not allow sufficient refinement, for modeling target behavior. For example, when the detection occurs after a preliminary target move, it is wise to handle target motion modeling. Itself depending on the target reactiveness capabilities, it follows that some final positions are more probable than other. To model this fact, we will simply introduce an up and down bounding on the probability associated with the target mixed strategy.

Similarly, it is also possible to define an up and down bounding on the resources sharing functions. Doing so involves a symmetrization of our problem. However, such bounding constraints on resources have a physical meaning. It implies a minimum and a maximum of resource affectation on each cell of the space search. Definitions have now to be clarified.

Definition: The placement of the target and the search are accomplished on a space E. Each element $x \in E$ is called a cell. The target mixed strategy is represented by a density function α defined on E. Function α is a variable of the problem. The summation of α on E is known and is denoted A_o. The following constraint then holds:

$$\int_E \alpha(x)\, dx = A_o \,.$$

Since α is a density probability, A_o generally equals 1. Two bounding functions α_1 and α_2 with property $\alpha_1 \leq \alpha_2$ are given. These functions are constants of the problem and yield a bounding constraint on the mixed target strategy:

$$\alpha_1 \leq \alpha \leq \alpha_2 \,.$$

The searcher pure strategy is represented by a resource sharing function φ defined on E. Function φ is also a variable of the problem. The total amount of resources ϕ_o is fixed, so that :

$$\int_E \varphi(x)\, dx = \phi_o \,.$$

Two bounding functions φ_1 and φ_2 with property $\varphi_1 \leq \varphi_2$ are given. These functions are constants of the problem and yield a bounding constraint on the pure search strategy:

$$\varphi_1 \leq \varphi \leq \varphi_2 \,.$$

For each cell x, a decreasing and convex non detection function p_x is defined. The value $p_x(\varphi(x))$ represents the conditional probability of non detection, when the target is located on cell x. The value of game for a couple of strategies (α, φ) is given by the averaged probability of non detection:

$$P_{nd}(\alpha, \varphi) = \int_E \alpha(x) p_x(\varphi(x))\, dx \ .$$

Again, since the game is convex, there is a couple of optimal strategies involving a mixed strategy for the target and a pure strategy for the searcher. The associated min-max optimization problem stands as follow:

Find:

$$\alpha_o = \arg\max_{\alpha} \min_{\varphi} \int_E \alpha(x) p_x(\varphi(x))\, dx$$

and

$$\varphi_o = \arg\min_{\varphi} \max_{\alpha} \int_E \alpha(x) p_x(\varphi(x))\, dx \ ,$$

under constraints:

$$\forall x \in E,\ \alpha_1(x) \leq \alpha(x) \leq \alpha_2(x) \ , \ \text{and} \ \int_E \alpha(x)\, dx = A_o \ ,$$

$$\forall x \in E,\ \varphi_1(x) \leq \varphi(x) \leq \varphi_2(x) \ , \ \text{and} \ \int_E \varphi(x)\, dx = \phi_o \ .$$

Summary of the problem setting:

- α_1: Lower bound for the target mixed strategy
- α_2: Upper bound for the target mixed strategy
- φ_1: Lower bound for the searcher strategy
- φ_2: Upper bound for the searcher strategy
- Constraints on the target mixed strategy:
  - ◇ $\alpha_1 \leq \alpha \leq \alpha_2$
  - ◇ $\int_E \alpha = A_o$
- Constraints on the searcher strategy:

- $\diamond$ $\varphi_1 \leq \varphi \leq \varphi_2$
- $\diamond$ $\int_E \varphi = \phi_o$

- $p_x(\varphi(x))$: Conditional probability of non detection, when resource $\varphi(x)$ is applied on cell x
  - $\diamond$ $p_x > 0$
  - $\diamond$ p_x decreases and $p'_x < 0$
  - $\diamond$ P_x is convex and $P''_x > 0$

Additive properties may be supposed in order to ensure the existence of solutions:

$$\int_E \alpha_1(x)\,dx \leq A_o \leq \int_E \alpha_2(x)\,dx \text{ and } \int_E \varphi_1(x)\,dx \leq \phi_o \leq \int_E \varphi_2(x)\,dx \;.$$

3 Optimality equations

Considering an optimal couple of strategies (α_o, φ_o) as a saddle point for the game value $P_{nd}(\alpha, \varphi)$, two optimality equations are obtained by variational means.

3.1 de Guenin's equation:

Since (α_o, φ_o) is a saddle point, it appears that:

$$\varphi_o \in \arg\min_{\varphi} P_{nd}(\alpha_o, \varphi) \;.$$

Constraints $\varphi_1 \leq \varphi \leq \varphi_2$ apply to the minimization. A result very similar to classical de Guenin's equation is thus obtained. More precisely, let $a \in E$ and $b \in E$ verifying $\varphi_o(a) > \varphi_1(a)$ and $\varphi_o(b) < \varphi_2(b)$. Let $dt > 0$ be a positive infinitesimal variation, and define a new sharing function $\tilde{\varphi}$ by:

$$\begin{cases} \tilde{\varphi}(a) = \varphi_o(a) - dt \text{ and } \tilde{\varphi}(b) = \varphi_o(b) + dt \;, \\ \tilde{\varphi}(x) = \varphi_o(x) \text{ for } x \neq a, b \;. \end{cases}$$

Then, by definition of a and b, constraint $\int_E \tilde{\varphi}(x)\,dx = \phi_o$ is also satisfied by the function $\tilde{\varphi}$. Thus, since φ_o is a minimizer, the probability increases, i.e. $P_{nd}(\alpha_o, \varphi_o) \leq P_{nd}(\alpha_o, \tilde{\varphi})$. Hence, $0 \leq -\alpha_o(a)p'_a(\varphi_o(a))dt + \alpha_o(b)p'_b(\varphi_o(b))dt$, by means of a first order derivation and a side to side simplification. Since $dt > 0$, the following inequality holds true:

$$\alpha_o(a)p'_a(\varphi_o(a)) \leq \alpha_o(b)p'_b(\varphi_o(b)) \ .$$

It is easy, then, to derive a weak optimality condition, i.e. the existence of a (negative) dual variable η such that:

$$\begin{cases} \varphi_1(x) < \varphi_o(x) < \varphi_2(x) \Rightarrow \alpha_o(x)p'_x(\varphi_o(x)) = \eta \ , \\ \varphi_o(x) = \varphi_1(x) \text{ or } \varphi_2(x) \text{ else .} \end{cases} \tag{6}$$

But this property is somewhat insufficient or poorly formulated for really defining φ_o. A more precise property will be proven. However, it requires further (but not restrictive) assumptions. First assumption is $\varphi_1 < \varphi_2$. This assumption is absolutely not restrictive, since for cells x verifying $\varphi_1(x) = \varphi_2(x)$, $\varphi_o(x)$ is defined by $\varphi_o(x) = \varphi_1(x) = \varphi_2(x)$. So, it is of no consequence not to consider these cases. We state also, that $\exists x,\ \varphi_o(x) > \varphi_1(x)$ and $\exists x,\ \varphi_o(x) < \varphi_2(x)$. This case is also no more restrictive, since otherwise, we would have $\forall x,\ \varphi_o(x) = \varphi_1(x)$ or $\forall x,\ \varphi_o(x) = \varphi_2(x)$, which are exactly equivalent to property $\phi_o = \int_E \varphi_1(x)\,dx$ or $\phi_o = \int_E \varphi_2(x)\,dx$ respectively. These specific cases are also directly checked, if necessary. Then, if all these assumptions are in use, the following property holds:

Proposition 1 *There exists a negative scalar η for which, the following alternative holds true for all $x \in E$:*

$$\begin{cases} \varphi_1(x) < \varphi_o(x) < \varphi_2(x) \Longrightarrow \alpha_o(x)p'_x(\varphi_o(x)) = \eta \ , \\ \varphi_o(x) = \varphi_1(x) \ \textit{or} \ \varphi_2(x) \ \textit{else} \ , \end{cases}$$

in accordance with the following discriminating equations:

$$\begin{cases} \alpha_o(x) > \dfrac{\eta}{p'_x(\varphi_1(x))} \Longrightarrow \varphi_o(x) > \varphi_1(x) \\ \alpha_o(x) < \dfrac{\eta}{p'_x(\varphi_2(x))} \Longrightarrow \varphi_o(x) < \varphi_2(x) \end{cases} \tag{7}$$

proof:

case a: The existence of a cell $b \in E$ such that $\varphi_1(b) < \varphi_o(b) < \varphi_2(b)$ is assumed for this case. Let $a \in E$ be a cell such that $\varphi_o(a) = \varphi_1(a)$ and $\alpha_o(a) > \frac{\eta}{p'_a(\varphi_1(a))}$. Let $dt > 0$ be a positive variation. Construct a perturbation $\tilde{\varphi}$ of φ_o defined by:

$$\begin{cases} \tilde{\varphi}(a) = \varphi_o(a) + dt \text{ and } \tilde{\varphi}(b) = \varphi_o(b) - dt \;, \\ \tilde{\varphi}(x) = \varphi_o(x) \text{ for } x \neq a, b \;. \end{cases}$$

The function $\tilde{\varphi}$ also satisfies to constraint $\int_E \tilde{\varphi}(x)\,dx = \phi_o$. Since φ_o is a minimizer of the mapping $\varphi \mapsto P_{nd}(\alpha_o, \varphi)$, the non-detection probability increases so that $P_{nd}(\alpha_o, \varphi_o) \leq P_{nd}(\alpha_o, \tilde{\varphi})$. First order derivation and side to side simplifications yield $0 \leq \alpha_o(a)p'_a(\varphi_o(a))dt - \alpha_o(b)p'_b(\varphi_o(b))dt$. Since $dt > 0$, equation $\alpha_o(a)p'_a(\varphi_o(a)) \geq \alpha_o(b)p'_b(\varphi_o(b))$ is obtained. Now, from hypothesis on b and equation (6), we have $\alpha(b)p'_b(\varphi_o(b)) = \eta$. A combination of the two previous results yields $\alpha_o(a) \leq \frac{\eta}{p'_a(\varphi_o(a))}$. This contradicts assumption on a. We have just refuted the existence of $x \in E$ such that $\varphi_o(x) = \varphi_1(x)$ and $\alpha_o(x) > \frac{\eta}{p'_x(\varphi_1(x))}$. Similarly, there is no $x \in E$ such that $\varphi_o(x) = \varphi_2(x)$ and $\alpha_o(x) < \frac{\eta}{p'_x(\varphi_2(x))}$. Thus, equations (7) are proven, whenever the existence of b is assumed.

case b : It is assumed now that there is no cell $b \in E$, such that $\varphi_1(b) < \varphi_o(b) < \varphi_2(b)$. In such case, variable η is not given by de Guenin's equation, and we will have to build it ourselves. Since $\exists x,\ \varphi_o(x) < \varphi_2(x)$ and $\exists x,\ \varphi_o(x) > \varphi_1(x)$, there is a cell a so that $\varphi_o(a) = \varphi_1(a)$ and a cell b such that $\varphi_o(b) = \varphi_2(b)$. Consider variation $dt > 0$ and perturbation $\tilde{\varphi}$ of φ_o defined by:

$$\begin{cases} \tilde{\varphi}(a) = \varphi_o(a) + dt \text{ and } \tilde{\varphi}(b) = \varphi_o(b) - dt \;, \\ \tilde{\varphi}(x) = \varphi_o(x) \text{ for } x \neq a, b \;. \end{cases}$$

Function $\tilde{\varphi}$ satisfy constraint $\int_E \tilde{\varphi}(x)\,dx = \phi_o$. Thus, the probability increases and consequently $P_{nd}(\alpha_o, \varphi_o) \leq P_{nd}(\alpha_o, \tilde{\varphi})$. After simplifications, the equation $\alpha_o(a)p'_a(\varphi_o(a)) \geq \alpha_o(b)p'_b(\varphi_o(b))$ is obtained. We have just proven:

$$\left.\begin{array}{r} \varphi_o(x) = \varphi_1(x) \\ \varphi_o(y) = \varphi_2(y) \end{array}\right\} \Rightarrow \alpha_o(x)p'_x(\varphi_o(x)) \geq \alpha_o(y)p'_y(\varphi_o(y)) \;.$$

Then, it becomes possible to define the dual variable η:

$$\exists\eta,\begin{cases}\varphi_o(x)=\varphi_1(x)\Longrightarrow\alpha_o(x)p'_x(\varphi_o(x))\geq\eta\ ,\\ \varphi_o(y)=\varphi_2(y)\Longrightarrow\alpha_o(y)p'_y(\varphi_o(y))\leq\eta\ .\end{cases}$$

Directly follows:

$$\exists\eta,\begin{cases}\varphi_o(x)=\varphi_1(x)\Longrightarrow\alpha_o(x)p'_x(\varphi_1(x))\geq\eta\ ,\\ \varphi_o(y)=\varphi_2(y)\Longrightarrow\alpha_o(y)p'_y(\varphi_2(y))\leq\eta\ .\end{cases}$$

Since $\varphi_1\leq\varphi_o\leq\varphi_2$, the properties $\varphi_o(x)=\varphi_1(x)$ and $\varphi_o(y)=\varphi_2(y)$ are equivalent to $\varphi_o(x)\leq\varphi_1(x)$ and $\varphi_o(y)\geq\varphi_2(y)$ respectively. At last:

$$\exists\eta,\begin{cases}\varphi_o(x)\leq\varphi_1(x)\Longrightarrow\alpha_o(x)p'_x(\varphi_1(x))\geq\eta\ ,\\ \varphi_o(y)\geq\varphi_2(y)\Longrightarrow\alpha_o(y)p'_y(\varphi_2(y))\leq\eta\ .\end{cases}$$

In other words, proposition 1 is also verified in this case.

□□□

Proposition 2 *There exists a negative scalar η for which, the following alternative holds true for all $x\in E$:*

$$\begin{cases}\varphi_1(x)<\varphi_o(x)<\varphi_2(x)\Longrightarrow\alpha_o(x)p'_x(\varphi_o(x))=\eta\ ,\\ \varphi_o(x)=\varphi_1(x)\ \textit{or}\ \varphi_2(x)\ \textit{else}\ ,\end{cases}$$

in accordance with the following discriminating equations:

$$\begin{cases}\alpha_o(x)>\dfrac{\eta}{p'_x(\varphi_1(x))}\Longleftrightarrow\varphi_o(x)>\varphi_1(x)\\ \alpha_o(x)<\dfrac{\eta}{p'_x(\varphi_2(x))}\Longleftrightarrow\varphi_o(x)<\varphi_2(x)\end{cases}\tag{8}$$

proof: The first parts are derived from proposition 1. In particular, implications (7) are proven. Now, assume $\varphi_o(x)>\varphi_1(x)$. It is possible that $\varphi_o(x)=\varphi_2(x)$ or that $\varphi_1(x)<\varphi_o(x)<\varphi_2(x)$.

case a: Make the hypothesis $\varphi_1(x)<\varphi_o(x)<\varphi_2(x)$. Then, de Guenin's

equation perfectly holds and $\alpha_o(x)p'_x(\varphi_o(x))=\eta$. Now, the function p' increases, $\eta<0$ and $\varphi_o(x)>\varphi_1(x)$. Hence $\alpha_o(x)=\frac{\eta}{p'_x(\varphi_o(x))}>\frac{\eta}{p'_x(\varphi_1(x))}$.
case b: Make the hypothesis $\varphi_o(x)=\varphi_2(x)$. Equation (7) yields the inequality $\alpha_o(x)\geq\frac{\eta}{p'_x(\varphi_2(x))}$. Now, p' increases, $\eta<0$ and $\varphi_2(x)>\varphi_1(x)$. Hence $\alpha_o(x)\geq\frac{\eta}{p'_x(\varphi_2(x))}>\frac{\eta}{p'_x(\varphi_1(x))}$.

The reversed implication $\varphi_o(x)>\varphi_1(x)\Rightarrow\alpha_o(x)>\frac{\eta}{p'_x(\varphi_1(x))}$ is proven now for both cases. Implication $\varphi_o(x)<\varphi_2(x)\Rightarrow\alpha_o(x)<\frac{\eta}{p'_x(\varphi_2(x))}$ similarly holds. Proposition 2 is now proven.

□□□

3.2 Constantness equation:

This part is almost similar to the preceding. First, it is noticed that:

$$\alpha_o\in\arg\max_{\alpha}P_{nd}(\alpha,\varphi_o)\ .$$

Constraint $\alpha_1\leq\alpha_o\leq\alpha_2$ applies to this minimization. Let $a\in E$ and $b\in E$ so that $\alpha_o(a)>\alpha_1(a)$ and $\alpha_o(b)<\alpha_2(b)$. Let $dt>0$ be a positive infinitesimal variation, and define a new mixed strategy $\tilde{\alpha}$ by:

$$\begin{cases}\tilde{\alpha}(a)=\alpha_o(a)-dt\text{ and }\tilde{\alpha}(b)=\alpha_o(b)+dt\ ,\\ \tilde{\alpha}(x)=\alpha_o(x)\text{ for }x\neq a,b\ .\end{cases}$$

Constraint $\int_E\tilde{\alpha}(x)\,dx=A_o$ still holds true. Thus, since α_o is a maximizer, the probability decreases, i.e. $P_{nd}(\alpha_o,\varphi_o)\geq P_{nd}(\tilde{\alpha},\varphi_o)$. Side to side simplifications of the inequality results in $-p_a(\varphi_o(a))dt+p_b(\varphi_o(b))dt\leq 0$. Since $dt>0$ we obtain:

$$p_a(\varphi_o(a))\geq p_b(\varphi_o(b))\ .$$

There is also a dual (positive) variable λ such that:

$$\begin{cases}\alpha_1(x)<\alpha_o(x)<\alpha_2(x)\Rightarrow p_x(\varphi_o(x))=\lambda\ ,\\ \alpha_o(x)=\alpha_1(x)\text{ or }\alpha_2(x)\text{ else}\ .\end{cases}\tag{9}$$

A more precise optimality equation will be proven now. Again, assumptions $\alpha_1 < \alpha_2$, $\exists x,\ \alpha_o(x) > \alpha_1(x)$ and $\exists x,\ \alpha_o(x) < \alpha_2(x)$ are made without loss of generality.

Proposition 3 *There exists a positive scalar λ for which, the following alternatives hold true for all $x \in E$:*

$$\begin{cases} \alpha_1(x) < \alpha_o(x) < \alpha_2(x) \Rightarrow p_x(\varphi_o(x)) = \lambda \ , \\ \alpha_o(x) = \alpha_1(x) \ \textit{or} \ \alpha_2(x) \ \textit{else} \ , \end{cases}$$

in accordance with the following discriminating equations:

$$\begin{cases} \varphi_o(x) < p_x^{-1}(\lambda) \Longrightarrow \alpha_o(x) > \alpha_1(x) \ , \\ \varphi_o(x) > p_x^{-1}(\lambda) \Longrightarrow \alpha_o(x) < \alpha_2(x) \ . \end{cases} \tag{10}$$

proof:

case a : The existence of a cell $b \in E$ such that $\alpha_1(b) < \alpha_o(b) < \alpha_2(b)$ is assumed for this case. Let $a \in E$ be a cell such that $\alpha_o(a) = \alpha_1(a)$ and $\varphi_o(a) < p_a^{-1}(\lambda)$. Let $dt > 0$ be a positive variation. Construct a perturbation $\tilde{\alpha}$ of α_o defined by:

$$\begin{cases} \tilde{\alpha}(a) = \alpha_o(a) + dt \text{ and } \tilde{\alpha}(b) = \alpha_o(b) - dt \ , \\ \tilde{\alpha}(x) = \alpha_o(x) \text{ for } x \neq a, b \ . \end{cases}$$

Constraint $\int_E \tilde{\alpha}(x)\,dx = A_o$ still holds true. Thus, the probability decreases and consequently $P_{nd}(\alpha_o, \varphi_o) \geq P_{nd}(\tilde{\alpha}, \varphi_o)$. After side to side simplifications and dt elimination, the equation $p_a(\varphi_o(a)) \leq p_b(\varphi_o(b))$ is obtained. Now, from hypothesis on b and equation (7) we have $p_b(\varphi_o(b)) = \lambda$. A combination of the two previous results yields $p_a(\varphi_o(a)) \leq \lambda$. This contradicts hypothesis on a. The existence of $x \in E$ such that $\alpha_o(x) = \alpha_1(x)$ and $\varphi_o(x) < p_x^{-1}(\lambda)$ has been refuted. Similarly, there is no $x \in E$ such that $\alpha_o(x) = \alpha_2(x)$ and $\varphi_o(x) > p_x^{-1}(\lambda)$. Now, $\alpha_1 \leq \alpha_o \leq \alpha_2$. Thus, equations (10) are proven, whenever b is located.

case b : It is assumed now that there is no cell $b \in E$, such that $\alpha_1(b) < \alpha_o(b) < \alpha_2(b)$. Since $\exists x,\ \alpha_o(x) < \alpha_2(x)$ and $\exists x,\ \alpha_o(x) > \alpha_1(x)$, there is both a cell a and a cell b such that $\alpha_o(a) = \alpha_1(a)$ and $\alpha_o(b) = \alpha_2(b)$. Consider variation $dt > 0$ and perturbation $\tilde{\alpha}$ of α_o defined by:

$$\begin{cases} \tilde{\alpha}(a) = \alpha_o(a) + dt \text{ and } \tilde{\alpha}(b) = \alpha_o(b) - dt \ , \\ \tilde{\alpha}(x) = \alpha_o(x) \text{ for } x \neq a, b \ . \end{cases}$$

Function $\tilde{\alpha}$ obeys to constraint $\int_E \tilde{\alpha}(x)\, dx = A_o$. The probability decreases so that $P_{nd}(\alpha_o, \varphi_o) \geq P_{nd}(\tilde{\alpha}, \varphi_o)$. Equation $p_a(\varphi_o(a)) \leq p_b(\varphi_o(b))$ is obtained after side to side simplifications and dt elimination. Thus, we have just proven:

$$\left.\begin{matrix} \alpha_o(x) = \alpha_1(x) \\ \alpha_o(y) = \alpha_2(y) \end{matrix}\right\} \Rightarrow p_x(\varphi_o(x)) \leq p_y(\varphi_o(y)) \ .$$

Then, it becomes possible to define the dual variable λ:

$$\exists \lambda, \begin{cases} \alpha_o(x) = \alpha_1(x) \Longrightarrow p_x(\varphi_o(x)) \leq \lambda \ , \\ \alpha_o(x) = \alpha_2(x) \Longrightarrow p_x(\varphi_o(x)) \geq \lambda \ . \end{cases}$$

Since $\alpha_1 \leq \alpha_o \leq \alpha_2$, the previous equation is equivalent to:

$$\exists \lambda, \begin{cases} \alpha_o(x) \leq \alpha_1(x) \Longrightarrow p_x(\varphi_o(x)) \leq \lambda \ , \\ \alpha_o(x) \geq \alpha_2(x) \Longrightarrow p_x(\varphi_o(x)) \geq \lambda \ . \end{cases}$$

Again, this property proves the existence of a dual variable λ satisfying proposition 3.

□□□

Proposition 4 *There exists a positive scalar λ for which, the following alternatives hold true for all $x \in E$:*

$$\begin{cases} \alpha_1(x) < \alpha_o(x) < \alpha_2(x) \Rightarrow p_x(\varphi_o(x)) = \lambda \ , \\ \alpha_o(x) = \alpha_1(x) \ \textit{or} \ \alpha_2(x) \ \textit{else} \ , \end{cases}$$

in accordance with the following discriminating equations:

$$\begin{cases} \varphi_o(x) < p_x^{-1}(\lambda) \Longrightarrow \alpha_o(x) = \alpha_2(x) \;, \\ \varphi_o(x) > p_x^{-1}(\lambda) \Longrightarrow \alpha_o(x) = \alpha_1(x) \;, \\ \alpha_o(x) > \alpha_1(x) \Longrightarrow \varphi_o(x) \leq p_x^{-1}(\lambda) \;, \\ \alpha_o(x) < \alpha_2(x) \Longrightarrow \varphi_o(x) \geq p_x^{-1}(\lambda) \;. \end{cases} \tag{11}$$

proof: The first parts are derived from proposition 3. In particular, implications (10) are proven, yielding:

$$\begin{cases} \varphi_o(x) < p_x^{-1}(\lambda) \Longrightarrow \alpha_o(x) = \alpha_2(x) \;, \\ \varphi_o(x) > p_x^{-1}(\lambda) \Longrightarrow \alpha_o(x) = \alpha_1(x) \;. \end{cases}$$

Now, assume $\alpha_o(x) > \alpha_1(x)$. Suppose $\varphi_o(x) > p_x^{-1}(\lambda)$. Then, from the implications (10), equation $\alpha_o(x) < \alpha_2(x)$ holds true. Since then the hypothesis $\alpha_1(x) < \alpha_o(x) < \alpha_2(x)$ is checked, it follows $p_x(\varphi_o(x)) = \lambda$, contradicting our assumption. Now, $\alpha_o(x) > \alpha_1(x) \Longrightarrow \varphi_o(x) \leq p_x^{-1}(\lambda)$ has just been proven. The implication $\alpha_o(x) < \alpha_2(x) \Longrightarrow \varphi_o(x) \geq p_x^{-1}(\lambda)$ is proven quite similarly.

□□□

3.3 Associated curves

The previous propositions 2 and 4 have an obvious geometric interpretation. For a given cell x, the optimal strategies (α_o, φ_o) are locally defined by the intersection of two curves H_η^x and Λ_λ^x. In other words, $(\alpha_o(x), \varphi_o(x)) \in H_\eta^x \cap \Lambda_\lambda^x$. These two curves are defined respectively from propositions 2 and 4:

$$\boxed{(\mathfrak{a}, \mathfrak{f}) \in H_\eta^x \Leftrightarrow \begin{cases} \mathfrak{a} \leq \frac{\eta}{p'_x(\varphi_1(x))} \Rightarrow \mathfrak{f} = \varphi_1(x) \\ \frac{\eta}{p'_x(\varphi_1(x))} < \mathfrak{a} < \frac{\eta}{p'_x(\varphi_2(x))} \Rightarrow \mathfrak{a}\, p'_x(\mathfrak{f}) = \eta \\ \mathfrak{a} \geq \frac{\eta}{p'_x(\varphi_2(x))} \Rightarrow \mathfrak{f} = \varphi_2(x) \end{cases}} \tag{12}$$

and

$$\boxed{(\mathfrak{a},\mathfrak{f})\in\Lambda_\lambda^x \Leftrightarrow \begin{cases} \mathfrak{f} < p_x^{-1}(\lambda) \Rightarrow \mathfrak{a} = \alpha_2(x) \\ \mathfrak{f} = p_x^{-1}(\lambda) \Rightarrow \mathfrak{a} \in [\alpha_1(x), \alpha_2(x)] \\ \mathfrak{f} > p_x^{-1}(\lambda) \Rightarrow \mathfrak{a} = \alpha_1(x) \end{cases}} \tag{13}$$

However, propositions 2 and 4 have a more precise meaning. There is a common choice of dual variables, which defines the whole optimal strategies as local intersection of the associated curves.

$$\exists \eta_o,\, \exists \lambda_o,\, \forall x \in E,\, (\alpha_o(x), \varphi_o(x)) \in H_{\eta_o}^x \cap \Lambda_{\lambda_o}^x\,. \tag{14}$$

We will use this viewpoint to develop an algorithmic resolution.

Graphical meaning: Since p_x is convex, p'_x is increasing and $\mathfrak{a} \mapsto {p'_x}^{-1}\left(\frac{\eta}{\mathfrak{a}}\right)$ is increasing ($\eta < 0$). Thus, H_η^x is flat ($\mathfrak{f} = \varphi_1(x)$) for $\mathfrak{a} \leq \frac{\eta}{p'_x(\varphi_1(x))}$, then becomes an increasing curve and is flat again ($\mathfrak{f} = \varphi_2(x)$) for $\mathfrak{a} \geq \frac{\eta}{p'_x(\varphi_2(x))}$. On the other hand, Λ_λ^x is vertically decreasing down to $p_x^{-1}(\lambda)$ for $\mathfrak{a} = \alpha_1(x)$. Then the curve becomes flat ($\mathfrak{f} = p_x^{-1}(\lambda)$) for $\alpha_1(x) \leq \mathfrak{a} \leq \alpha_2(x)$ and, at last, the curve is vertically decreasing down from $p_x^{-1}(\lambda)$ for $\mathfrak{a} = \alpha_2(x)$. These two curves are schematized in figure 1.

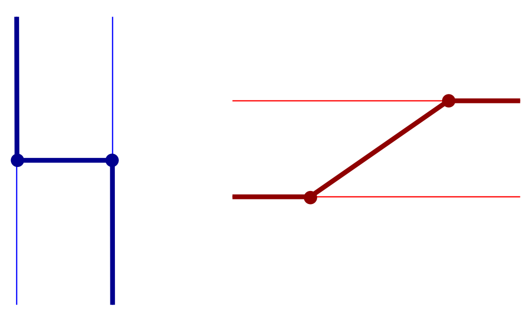


Figure 1: Curves Λ_λ^x and H_η^x.

Now, the flatness of the curves has a consequence. Intersections may be a segment and not a point, as it is shown in figure 2. Non uniqueness of the

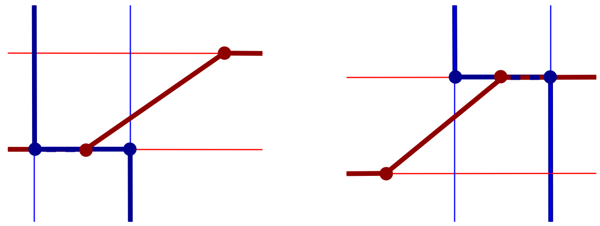


Figure 2: Undefined cases

intersection is related to possibly indetermined cases for the optimality equations. In these cases it is necessary to use the constraint equations $\int_E \alpha = A_o$ and $\int_E \varphi = \phi_o$. But it may happen that several solutions are optimal. For instance, assume an optimal solution (α_o, φ_o), with two cells a and b, verifying for both $x = a$ and $x = b$:

$$p_x^{-1}(\lambda_o) = \varphi_1(x) \text{ and } \alpha_2(x) \leq \frac{\eta}{p'_x(\varphi_1(x))} \ .$$

These cells are in the undefined status represented in figure 3. In particular,

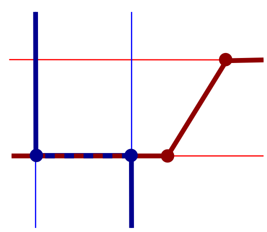


Figure 3: Example

for these cases, the available choice of $\alpha(x)$ is given by a set $[\alpha_1(x), \alpha_2(x)]$. Assume that our particular optimal solution verifies $\alpha_1(a) < \alpha_o(a) < \alpha_2(a)$ and $\alpha_1(b) < \alpha_o(b) < \alpha_2(b)$. Let dt be an infinitesimal variation and define $\tilde{\alpha}_o$ by:

$$\begin{cases} \tilde{\alpha}_o(a) = \alpha_o(a) + dt \text{ and } \tilde{\alpha}_o(b) = \alpha_o(b) - dt \ , \\ \tilde{\alpha}_o(x) = \alpha_o(x) \text{ for } x \neq a, b \ . \end{cases}$$

The couple $(\tilde{\alpha}_o, \varphi_o)$ still verifies the optimality equations and the constraints of the problem. It appears that $(\tilde{\alpha}_o, \varphi_o)$ is still an optimal solution of the game.

3.4 Inverting the optimality equations

Optimality equations have been derived in the previous sections (3.1, 3.2 and 3.3). In order to develop a practical algorithm, the next step is to invert them.

Mapping $(\eta, \lambda) \mapsto (\alpha^{\eta\lambda}, \varphi^{\eta\lambda})$**:** The previous remarks permit us to build a mapping from the dual variable (η, λ) to the associated strategies $(\alpha^{\eta\lambda}, \varphi^{\eta\lambda})$, which inverts the optimality equations. As seen previously, this mapping may point to more than one strategy. What we have to define is a multivalued function. Now, the curves shape induces that $H^x_{\eta_o} \cap \Lambda^x_{\lambda_o}$ is always an horizontal closed interval. In other word, the mapping is $1:1$ for $\varphi^{\eta\lambda}$; while, for each $\alpha^{\eta\lambda}(x)$, it is given by a continuum from a minimum value $\alpha^{\eta\lambda}_{min}(x)$ to a maximum value $\alpha^{\eta\lambda}_{max}(x)$, even if we have generally $\alpha^{\eta\lambda}_{min}(x) = \alpha^{\eta\lambda}_{max}(x)$. In fact, because of the middle flatness of Λ^x_λ and the two extremal flatness of H^x_η, **there is at most two** λ **such that** $\alpha^{\eta\lambda}_{min}(x) < \alpha^{\eta\lambda}_{max}(x)$. Now, the following mapping may be defined, for the solutions associated to the optimality constraints on (η, λ) :

$$(\eta, \lambda) \longmapsto \left[\alpha^{\eta\lambda}_{min}, \alpha^{\eta\lambda}_{max}\right] \times \left\{\varphi^{\eta\lambda}\right\} \ .$$

The crucial point, is that $\alpha^{\eta\lambda}_{min}$, $\alpha^{\eta\lambda}_{max}$ and $\varphi^{\eta\lambda}$ are simply and entirely defined and computable by means of the problem data. However, we shall not give an explicit definition of these functions in the main part of this paper, since a lot of case checking is required. Reader should refer to apprendix A for more details.

Knowing $\alpha^{\eta\lambda}_{min}$, $\alpha^{\eta\lambda}_{max}$ and $\varphi^{\eta\lambda}$ it is useful to define the following global values:

$$\begin{cases} \phi^{\eta\lambda} = \displaystyle\int_E \varphi^{\eta\lambda}(x)\,dx \\ A^{\eta\lambda}_{min} = \displaystyle\int_E \alpha^{\eta\lambda}_{min}(x)\,dx \\ A^{\eta\lambda}_{max} = \displaystyle\int_E \alpha^{\eta\lambda}_{max}(x)\,dx \end{cases}$$

Values $\phi^{\eta\lambda}$, $A^{\eta\lambda}_{min}$ and $A^{\eta\lambda}_{max}$ will be of constant use in the development of our algorithm.

Variation of $\phi^{\eta\lambda}$, $A^{\eta\lambda}_{min}$ and $A^{\eta\lambda}_{max}$: Our interest now focuses on the variation of $\phi^{\eta\lambda}$, $A^{\eta\lambda}_{min}$ and $A^{\eta\lambda}_{max}$ according to the variables η and λ. First, it appears that an increase of η produces an up swelling (associated to a left shifting) of the curve H^x_η, more precisely:

$$\eta_1 < \eta_2 \Rightarrow \left[\forall x,\ \forall \mathfrak{a},\ \left. \begin{array}{c} (\mathfrak{a}, \mathfrak{f}_1) \in H^x_{\eta_1} \\ (\mathfrak{a}, \mathfrak{f}_2) \in H^x_{\eta_2} \end{array} \right\} \Rightarrow \mathfrak{f}_2 \geq \mathfrak{f}_1 \right] \tag{15}$$

proof: Assume $\eta_1 < \eta_2$. Let $x \in E$. For $\mathfrak{a}$ be given, define $\mathfrak{f}_1$ and $\mathfrak{f}_2$ by $(\mathfrak{a}, \mathfrak{f}_1) \in H^x_{\eta_1}$ and $(\mathfrak{a}, \mathfrak{f}_2) \in H^x_{\eta_2}$. Three cases are considered. In the first case, suppose $\mathfrak{a} \leq \frac{\eta_1}{p'_x(\varphi_1(x))}$. Then (refer to definition (12)), $\mathfrak{f}_1 = \varphi_1(x)$, and thus, $\mathfrak{f}_1 \leq \mathfrak{f}_2$. In the second case, suppose $\mathfrak{a} \geq \frac{\eta_2}{p'_x(\varphi_2(x))}$. Then, $\mathfrak{f}_2 = \varphi_2(x)$, and thus, $\mathfrak{f}_2 \geq \mathfrak{f}_1$. Now let us consider the remaining case $\mathfrak{a} \in \left] \frac{\eta_1}{p'_x(\varphi_1(x))}, \frac{\eta_2}{p'_x(\varphi_2(x))} \right[$. Hypothesis $\eta_1 < \eta_2$ yields $\frac{\eta_1}{p'_x(\varphi_1(x))} > \frac{\eta_2}{p'_x(\varphi_1(x))}$ and $\frac{\eta_1}{p'_x(\varphi_2(x))} > \frac{\eta_2}{p'_x(\varphi_2(x))}$, so that are verified both $\mathfrak{a} \in \left] \frac{\eta_1}{p'_x(\varphi_1(x))}, \frac{\eta_1}{p'_x(\varphi_2(x))} \right[$ and $\mathfrak{a} \in \left] \frac{\eta_2}{p'_x(\varphi_1(x))}, \frac{\eta_2}{p'_x(\varphi_2(x))} \right[$. It follows that $\mathfrak{f}_1 = {p'_x}^{-1}\left(\frac{\eta_1}{\mathfrak{a}}\right)$ and $\mathfrak{f}_2 = {p'_x}^{-1}\left(\frac{\eta_2}{\mathfrak{a}}\right)$. Now, $\eta_1 < \eta_2$ and ${p'_x}^{-1}$ is increasing, hence $\mathfrak{f}_1 \leq \mathfrak{f}_2$. Ending the proof of (15).

□□□

The increase of η yields also an up-left move of the intersection $H_\eta^x \cap \Lambda_\lambda^x$:

$$\eta_1 < \eta_2 \Longrightarrow \begin{cases} \alpha_{min}^{\eta_1\lambda}(x) \geq \alpha_{min}^{\eta_2\lambda}(x) \\ \alpha_{max}^{\eta_1\lambda}(x) \geq \alpha_{max}^{\eta_2\lambda}(x) \\ \varphi^{\eta_1\lambda}(x) \leq \varphi^{\eta_2\lambda}(x) \end{cases} \tag{16}$$

This property is a direct consequence of the following lemma.

Lemma 1 *Let $\eta_2 > \eta_1$. The two following implications hold:*

$$\forall(\mathfrak{a}_1, \mathfrak{f}_1) \in H_{\eta_1}^x \cap \Lambda_\lambda^x\,,\ \exists(\mathfrak{a}_2, \mathfrak{f}_2) \in H_{\eta_2}^x \cap \Lambda_\lambda^x\,,\ \mathfrak{f}_2 \geq \mathfrak{f}_1 \textit{ and } \mathfrak{a}_2 \leq \mathfrak{a}_1\ .$$

$$\forall(\mathfrak{a}_2, \mathfrak{f}_2) \in H_{\eta_2}^x \cap \Lambda_\lambda^x\,,\ \exists(\mathfrak{a}_1, \mathfrak{f}_1) \in H_{\eta_1}^x \cap \Lambda_\lambda^x\,,\ \mathfrak{f}_1 \leq \mathfrak{f}_2 \textit{ and } \mathfrak{a}_1 \geq \mathfrak{a}_2\ .$$

proof of lemma: Only the first implication is proven. The second implication holds similarly. Let $(\mathfrak{a}_1, \mathfrak{f}_1) \in H_{\eta_1}^x \cap \Lambda_\lambda^x$. Define $\tilde{\mathfrak{f}}_1$ by $(\mathfrak{a}_1, \tilde{\mathfrak{f}}_1) \in H_{\eta_2}^x$. From property (15), it follows $\tilde{\mathfrak{f}}_1 \geq \mathfrak{f}_1$. since Λ_λ^x is flat or vertically decreasing, the point $(\mathfrak{a}_1, \tilde{\mathfrak{f}}_1)$ is either in the curve Λ_λ^x or above this curve. When first case holds, i.e. $(\mathfrak{a}_1, \tilde{\mathfrak{f}}_1) \in \Lambda_\lambda^x$, then $(\mathfrak{a}_1, \tilde{\mathfrak{f}}_1) \in H_{\eta_2}^x \cap \Lambda_\lambda^x$. Thus, it suffices to take $\mathfrak{a}_2 = \mathfrak{a}_1$ and $\mathfrak{f}_2 = \tilde{\mathfrak{f}}_1$, so as to fulfill the implication. Now, assume that second case holds, i.e. point $(\mathfrak{a}_1, \tilde{\mathfrak{f}}_1)$ is above the curve Λ_λ^x. Since Λ_λ^x is a decreasing or flat curve, the point $(\mathfrak{a}, \mathfrak{f})$ is above Λ_λ^x whenever $\mathfrak{a} \geq \mathfrak{a}_1$ and $\mathfrak{f} \geq \tilde{\mathfrak{f}}_1$. Now $H_{\eta_2}^x$ is an increasing or flat curve. Thus, for $\mathfrak{a} \geq \mathfrak{a}_1$, property $(\mathfrak{a}, \mathfrak{f}) \in H_{\eta_2}^x$ yields $\mathfrak{f} \geq \tilde{\mathfrak{f}}_1$ and consequently $(\mathfrak{a}, \mathfrak{f})$ is then above Λ_λ^x. It follows that $\mathfrak{a}_2 \leq \mathfrak{a}_1$ whenever $(\mathfrak{a}_2, \mathfrak{f}_2) \in H_{\eta_2}^x \cap \Lambda_\lambda^x$. Since Λ_λ^x is a decreasing or flat curve, also holds $\mathfrak{f}_2 \geq \mathfrak{f}_1$. Consequently, the implication holds again.

□□□

An increase of λ produces similarly a left swelling (associated to a down shifting) of curve Λ_λ^x:

$$\lambda_1 < \lambda_2 \Rightarrow \left[\forall x,\ \forall \mathfrak{f},\ \begin{cases} (\mathfrak{a}_1, \mathfrak{f}) \in \Lambda_{\lambda_1}^x \Rightarrow \mathfrak{a}_1 = \alpha_2(x) \\ \qquad\qquad \text{or} \\ (\mathfrak{a}_2, \mathfrak{f}) \in \Lambda_{\lambda_2}^x \Rightarrow \mathfrak{a}_2 = \alpha_1(x) \end{cases} \right] \tag{17}$$

proof: Assume $\lambda_1 < \lambda_2$. Let $x \in E$. For $\mathfrak{f}$ be given, define $\mathfrak{a}_1$ and $\mathfrak{a}_2$ by $(\mathfrak{a}_1, \mathfrak{f}) \in \Lambda^x_{\lambda_1}$ and $(\mathfrak{a}_2, \mathfrak{f}) \in \Lambda^x_{\lambda_2}$. Two cases are considered. In the first case, suppose $\mathfrak{f} < p_x^{-1}(\lambda_1)$. Then (refer to definition (13)), $\mathfrak{a}_1 = \alpha_2(x)$. In the second case, suppose $\mathfrak{f} \geq p_x^{-1}(\lambda_1)$. Function p_x^{-1} decreases and $\lambda_1 < \lambda_2$. Thus $p_x^{-1}(\lambda_1) > p_x^{-1}(\lambda_2)$. Hence $\mathfrak{f} > p_x^{-1}(\lambda_2)$. Then $\mathfrak{a}_2 = \alpha_1(x)$. Property (17) is thus proven.

□□□

Property (18) holds as a direct corollary of (17):

$$\lambda_1 < \lambda_2 \Rightarrow \left[\forall x,\, \forall \mathfrak{f},\ \left.\begin{array}{c} (\mathfrak{a}_1, \mathfrak{f}) \in \Lambda^x_{\lambda_1} \\ (\mathfrak{a}_2, \mathfrak{f}) \in \Lambda^x_{\lambda_2} \end{array}\right\} \Rightarrow \mathfrak{a}_2 \leq \mathfrak{a}_1 \right] \tag{18}$$

The intersection $H^x_\eta \cap \Lambda^x_\lambda$ moves down-left. However, the intersection variation is somewhat *sharper* here:

$$\lambda_1 < \lambda_2 \Longrightarrow \left\{ \begin{array}{l} \alpha^{\eta\lambda_1}_{min}(x) \geq \alpha^{\eta\lambda_2}_{max}(x) \\ \varphi^{\eta\lambda_1}(x) \geq \varphi^{\eta\lambda_2}(x) \end{array} \right. \tag{19}$$

proof: Let $\lambda_1 < \lambda_2$ and $x \in E$. Let $(\mathfrak{a}_1, \mathfrak{f}_1) \in H^x_\eta \cap \Lambda^x_{\lambda_1}$. Define $\tilde{\mathfrak{a}}_1$ a point such that $(\tilde{\mathfrak{a}}_1, \mathfrak{f}_1) \in \Lambda^x_{\lambda_2}$. From property (18), it follows $\tilde{\mathfrak{a}}_1 \leq \mathfrak{a}_1$. Since H^x_η is a flat or increasing curve, point $(\tilde{\mathfrak{a}}_1, \mathfrak{f}_1)$ is either in the curve H^x_η or at the left outer of this curve.

First case: $(\tilde{\mathfrak{a}}_1, \mathfrak{f}_1) \in H^x_\eta$. Then also holds $(\tilde{\mathfrak{a}}_1, \mathfrak{f}_1) \in H^x_\eta \cap \Lambda^x_{\lambda_2}$. Thus equation $\varphi^{\eta\lambda_2}(x) = \varphi^{\eta\lambda_1}(x)$ is deduced. Now, the property (17) yields either the implication $(\mathfrak{a}_1, \varphi^{\eta\lambda_1}(x)) \in \Lambda^x_{\lambda_1} \Rightarrow \mathfrak{a}_1 = \alpha_2(x)$ or the implication $(\mathfrak{a}_2, \varphi^{\eta\lambda_2}(x)) \in \Lambda^x_{\lambda_2} \Rightarrow \mathfrak{a}_2 = \alpha_1(x)$. In other words, $\alpha^{\eta\lambda_1}_{min}(x) \geq \alpha^{\eta\lambda_2}_{max}(x)$.

Second case: Point $(\tilde{\mathfrak{a}}_1, \mathfrak{f}_1)$ is at left outer of curve H^x_η. Since H^x_η is an increasing curve, points $(\mathfrak{a}, \mathfrak{f})$ such that $\mathfrak{a} \leq \tilde{\mathfrak{a}}_1$ and $\mathfrak{f} \geq \mathfrak{f}_1$ are still at left outer of this curve. Now, $\Lambda^x_{\lambda_2}$ is a decreasing or flat curve. Thus, for $\mathfrak{f} \geq \mathfrak{f}_1$, property $(\mathfrak{a}, \mathfrak{f}) \in \Lambda^x_{\lambda_2}$ yields $\mathfrak{a} \leq \tilde{\mathfrak{a}}_1$ and, consequently, $(\mathfrak{a}, \mathfrak{f})$ is then at left outer of H^x_η. It follows that $\mathfrak{f}_2 < \mathfrak{f}_1$, whenever $(\mathfrak{a}_2, \mathfrak{f}_2) \in H^x_\eta \cap \Lambda^x_{\lambda_2}$. Hence $\varphi^{\eta\lambda_2}(x) < \varphi^{\eta\lambda_1}(x)$. Now, H^x_η is an increasing or flat curve, and moreover, this curve never increases vertically. This signifies:

$$\left.\begin{array}{r} (\mathfrak{a}_1, \mathfrak{f}_1) \in H^x_\eta \\ (\mathfrak{a}_2, \mathfrak{f}_2) \in H^x_\eta \\ \mathfrak{f}_1 > \mathfrak{f}_2 \end{array}\right\} \Longrightarrow \mathfrak{a}_1 > \mathfrak{a}_2 \, .$$

Thus, since $\varphi^{\eta\lambda_2}(x) < \varphi^{\eta\lambda_1}(x)$, it follows that $\alpha_{max}^{\eta\lambda_2}(x) < \alpha_{min}^{\eta\lambda_1}(x)$. Property (19) is then proven.

□□□

Global results are derived:

$$\forall\lambda,\ \eta_1 < \eta_2 \Rightarrow \begin{cases} A_{min}^{\eta_1\lambda} \geq A_{min}^{\eta_2\lambda} \\ A_{max}^{\eta_1\lambda} \geq A_{max}^{\eta_2\lambda} \\ \phi^{\eta_1\lambda} \leq \phi^{\eta_2\lambda} \end{cases} \tag{20}$$

and

$$\forall\eta,\ \lambda_1 < \lambda_2 \Rightarrow \begin{cases} A_{min}^{\eta\lambda_1} \geq A_{max}^{\eta\lambda_2} \\ \phi^{\eta\lambda_1} \geq \phi^{\eta\lambda_2} \end{cases} \tag{21}$$

Implicit definition of $\eta(\lambda)$: Let λ be fixed. In this situation, the curve Λ_λ^x is also fixed. Then, what happen, when η is varying? Define:

$$\eta_{min} = \min_x \Big(\alpha_2(x) p'_x(\varphi_1(x))\Big)\ ,$$

and

$$\eta_{max} = \max_x \Big(\alpha_1(x) p'_x(\varphi_2(x))\Big)\ .$$

These two values form a bound for the dual variables η. In fact, inverting de Guenin optimality equation, i.e. $(\alpha, \varphi) \in H_\eta$ thanks to definition (12), yields:

$$\forall\lambda, \begin{cases} \eta \leq \eta_{min} \Longrightarrow \varphi^{\eta\lambda} = \varphi_1 \\ \eta \geq \eta_{max} \Longrightarrow \varphi^{\eta\lambda} = \varphi_2 \end{cases}\ .$$

So, for $\eta \leq \eta_{min}$ or $\eta \geq \eta_{max}$, function $\varphi^{\eta\lambda}$ is entirely defined and independent of the dual variable λ. More precisely, there are two possible configurations, say $\eta \leq \eta_{min}$ and $\varphi^{\eta\lambda} = \varphi_1$, or $\eta \geq \eta_{max}$ and $\varphi^{\eta\lambda} = \varphi_2$. Now, consequently to the definition (13) of curve Λ_λ), $\alpha_{min}^{\eta\lambda}$ and $\alpha_{max}^{\eta\lambda}$ are entirely defined by the choice of λ and of the configuration:

$$\forall\lambda, \begin{cases} [\eta_1 \leq \eta_{min} \text{ and } \eta_2 \leq \eta_{min}] \Longrightarrow (\varphi^{\eta_1\lambda}, \alpha_{min}^{\eta_1\lambda}, \alpha_{max}^{\eta_1\lambda}) = (\varphi^{\eta_2\lambda}, \alpha_{min}^{\eta_2\lambda}, \alpha_{max}^{\eta_2\lambda}) \\ [\eta_1 \geq \eta_{max} \text{ and } \eta_2 \geq \eta_{max}] \Longrightarrow (\varphi^{\eta_1\lambda}, \alpha_{min}^{\eta_1\lambda}, \alpha_{max}^{\eta_1\lambda}) = (\varphi^{\eta_2\lambda}, \alpha_{min}^{\eta_2\lambda}, \alpha_{max}^{\eta_2\lambda}) \end{cases}\ .$$

In other word, every possible configurations for the game may be represented by a dual variable $\eta \in [\eta_{min}, \eta_{max}]$ and it is this range, we will consider from now on. Extremal global resource values are given then by:

$$\phi^{\eta_{min}} = \phi^{\eta_{min}\lambda} = \int_E \varphi_1(x)\,dx \text{ and } \phi^{\eta_{max}} = \phi^{\eta_{max}\lambda} = \int_E \varphi_2(x)\,dx \;.$$

To prove the incoming result, the following lemma is needed:

Lemma 2 *Dual parameters η and λ be given, let $d\eta$ be an infinitesimal variation of η. Then, for any given cell $x \in E$, the variation of $H_\eta^x \cap \Lambda_\lambda^x$ is infinitesimal, i.e.:*

$$\left.\begin{array}{r}(\mathfrak{a},\mathfrak{f}) \in H_\eta^x \cap \Lambda_\lambda^x \\ d\eta < 0\end{array}\right\} \Rightarrow H_{\eta+d\eta}^x \cap \Lambda_\lambda^x \cap \left[\mathfrak{a}, \mathfrak{a} + \frac{2d\eta}{p'_x(\varphi_2(x))}\right] \times \left[\mathfrak{f} + \frac{2d\eta}{\alpha_1(x)m''_x}, \mathfrak{f}\right] \neq \emptyset \;,$$

and

$$\left.\begin{array}{r}d\eta > 0 \\ (\mathfrak{a},\mathfrak{f}) \in H_\eta^x \cap \Lambda_\lambda^x\end{array}\right\} \Rightarrow H_{\eta+d\eta}^x \cap \Lambda_\lambda^x \cap \left[\mathfrak{a} + \frac{2d\eta}{p'_x(\varphi_2(x))}, \mathfrak{a}\right] \times \left[\mathfrak{f}, \mathfrak{f} + \frac{2d\eta}{\alpha_1(x)m''_x}\right] \neq \emptyset \;, \tag{22}$$

where $m''_x = \min_{\mathfrak{f} \in [\varphi_1(x), \varphi_2(x)]} p''_x(\mathfrak{f})$.

proof: Refer to Apendix B.

□□□

Now, λ being fixed, the function $\eta \mapsto \phi^{\eta\lambda}$ appears to be a continuous and increasing (or flat) mapping from interval $[\eta_{min}, \eta_{max}]$ onto interval $[\phi^{\eta_{min}}, \phi^{\eta_{max}}]$.

proof: Let $d\eta$ be a variation of η. Assume $d\eta > 0$. Locally to a particular cell $x \in E$, previous lemma says $\varphi^{\eta\lambda}(x) \leq \varphi^{\eta+d\eta\lambda}(x) \leq \varphi^{\eta\lambda}(x) + \frac{2d\eta}{\alpha_1(x)M''_x}$. A summation of $\varphi^{\eta\lambda}$ yields so $\phi^{\eta\lambda} \leq \phi^{\eta+d\eta\lambda} \leq \phi^{\eta\lambda} + \left(\int_E \frac{2dx}{\alpha_1(x)M''_x}\right) d\eta$. Thus, mapping $\eta \mapsto \phi^{\eta\lambda}$ is continuous and increasing (or flat). As a consequence, it is also surjective.

□□□

It follows that every $\phi \in \left[\int_E \varphi_1, \int_E \varphi_2\right]$ admits a non empty connected set of antecedents. It is in particular true for ϕ_o. The set of antecedents is often reduced to one element, when mapping $\eta \mapsto \phi^{\eta\lambda}$ is increasing, otherwise it is an interval when mapping $\eta \mapsto \phi^{\eta\lambda}$ is flat. This remark implicitly defines the bounds $\eta_{min}(\lambda) \in [\eta_{min}, \eta_{max}]$ and $\eta_{max}(\lambda) \in [\eta_{min}, \eta_{max}]$ of parameters η, verifying de Guenin's equations according to λ:

$$\forall \eta \in [\eta_{min}, \eta_{max}]\,,\; \phi^{\eta\lambda} = \phi_o \Leftrightarrow \eta \in [\eta_{min}(\lambda), \eta_{max}(\lambda)]\ .$$

Assume now $\lambda_1 < \lambda_2$. Property (21) yields $\phi^{\eta_{min}(\lambda_2)\lambda_1} \geq \phi^{\eta_{min}(\lambda_2)\lambda_2}$. Since by definition $\phi^{\eta_{min}(\lambda_2)\lambda_2} = \phi_o = \phi^{\eta_{min}(\lambda_1)\lambda_1}$, property $\phi^{\eta_{min}(\lambda_2)\lambda_1} \geq \phi^{\eta_{min}(\lambda_1)\lambda_1}$ holds. Now, imagine $\eta_{min}(\lambda_2) < \eta_{min}(\lambda_1)$. The property (20) yields the reversed inequality $\phi^{\eta_{min}(\lambda_2)\lambda_1} \leq \phi^{\eta_{min}(\lambda_1)\lambda_1}$. It follows $\phi^{\eta_{min}(\lambda_2)\lambda_1} = \phi^{\eta_{min}(\lambda_1)\lambda_1} = \phi_o$. Thus $\eta_{min}(\lambda_2) \geq \eta_{min}(\lambda_1)$, by minimal definition of $\eta_{min}(\lambda_1)$. Hence a contradiction of hypothesis $\eta_{min}(\lambda_2) < \eta_{min}(\lambda_1)$. So, property $\eta_{min}(\lambda_2) \geq \eta_{min}(\lambda_1)$ holds true, and similarly, property $\eta_{max}(\lambda_2) \geq \eta_{max}(\lambda_1)$ may be proven. Variations of $\eta_{min}(\lambda)$ and $\eta_{max}(\lambda)$ are monotonic:

$$\lambda_1 < \lambda_2 \Longrightarrow \begin{cases} \eta_{min}(\lambda_1) \leq \eta_{min}(\lambda_2) \\ \eta_{max}(\lambda_1) \leq \eta_{max}(\lambda_2) \end{cases}$$

Now, $A^{\eta\lambda}_{min}$ and $A^{\eta\lambda}_{max}$ are decreasing for both η and λ. Thus previous results yield:

$$\lambda_1 < \lambda_2 \Longrightarrow \begin{cases} A_{min}^{\eta_{min}(\lambda_1)\lambda_1} \geq A_{min}^{\eta_{min}(\lambda_2)\lambda_2} \\ A_{max}^{\eta_{min}(\lambda_1)\lambda_1} \geq A_{max}^{\eta_{min}(\lambda_2)\lambda_2} \\ A_{min}^{\eta_{max}(\lambda_1)\lambda_1} \geq A_{min}^{\eta_{max}(\lambda_2)\lambda_2} \\ A_{max}^{\eta_{max}(\lambda_1)\lambda_1} \geq A_{max}^{\eta_{max}(\lambda_2)\lambda_2} \end{cases} \tag{23}$$

4 Algorithm

The previous properties are a guideline for developing our algorithm. Since optimality equations are almost invertible and signs of variation are fixed for $A^{\eta\lambda}_{min}$, $A^{\eta\lambda}_{max}$ and $\phi^{\eta\lambda}$, bi-sectional methods were chosen. Our algorithm is made of three parts. First part find the optimal dual parameter λ_o. The second part sharpens the convergence and renders more precise some subdefinitions, by calibrating the optimal dual parameter η_o. At this point, convergence is almost achieved. The last part makes a final calibration of α, so as to equalize to A_o and to reduce some indetermination.

Computing λ_o and η_o: The first ingredient is to build up the procedure, which defines $\eta(\lambda)$, that is, which computes $\eta_{min}(\lambda)$ and $\eta_{max}(\lambda)$. Thanks to the increaseness property associated with the definition of $\eta_{min}(\lambda)$ and $\eta_{max}(\lambda)$, two bi-sectional processes around ϕ_o are in use to compute $\eta_{min}(\lambda)$ and $\eta_{max}(\lambda)$. Then, the main part of the process will consist in finding λ, such that $A_o \in \left[A_{min}^{\eta_{max}(\lambda)\lambda}, A_{max}^{\eta_{min}(\lambda)\lambda}\right]$. Thanks to the increaseness evoked in property (23), this is done again by a bi-sectional process. However, this process will call the procedure for $\eta_{min}(\lambda)$ and $\eta_{max}(\lambda)$ computation, constituting in fact a double bi-sectional procedure. This procedure yields as a result the optimal dual variable λ_o. It is noteworthy that for $\eta \in [\eta_{min}(\lambda_o), \eta_{max}(\lambda_o)]$, $\phi^{\eta\lambda_o} = \phi_o$ and we will not have to care about the constraint on ϕ_o, now. Otherwise, since $A_o \in \left[A_{min}^{\eta_{max}(\lambda_o)\lambda_o}, A_{max}^{\eta_{min}(\lambda_o)\lambda_o}\right]$, there exists $\eta \in [\eta_{min}(\lambda_o), \eta_{max}(\lambda_o)]$ such that $A_o \in \left[A_{min}^{\eta\lambda_o}, A_{max}^{\eta\lambda_o}\right]$. This element η will be our optimal dual variable η_o. To compute it, a bi-sectional process is again instrumental, because of the constant sign variations of $A_{min}^{\eta\lambda_o}$ and $A_{max}^{\eta\lambda_o}$ (refer to property (20)). The whole process is summed up below:

i. Find λ_o such that $A_o \in \left[A_{min}^{\eta_{max}(\lambda_o)\lambda_o}, A_{max}^{\eta_{min}(\lambda_o)\lambda_o}\right]$; *do it by means of a bi-sectional process; a sub-procedure is used to compute* $\eta_{min}(\lambda)$ *and* $\eta_{max}(\lambda)$,

ii. Find η_o, element of $[\eta_{min}(\lambda_o), \eta_{max}(\lambda_o)]$, such that $A_o \in \left[A_{min}^{\eta_o\lambda_o}, A_{max}^{\eta_o\lambda_o}\right]$; *do it by means of a bi-sectional process.*

sub-procedure: Compute $\eta_{min}(\lambda)$ and $\eta_{max}(\lambda)$ by means of a bi-sectional process.

Finalization: Now, η_o and λ_o are found. Function φ_o is entirely defined by $\varphi^{\eta_o\lambda_o}$. However, there could be some indetermination for α_o, in particular when $A^{\eta_o\lambda_o}_{min} < A^{\eta_o\lambda_o}_{max}$. Now, definitions of $A^{\eta\lambda}_{min}$ and $A^{\eta\lambda}_{max}$ say $A^{\eta\lambda}_{min} = \int_E \alpha^{\eta\lambda}_{min}$ and $A^{\eta\lambda}_{max} = \int_E \alpha^{\eta\lambda}_{max}$. Thus, a candidate α_o, such that $\int_E \alpha_o = A^o$, may be defined as the barycenter of $\alpha^{\eta_o\lambda_o}_{min}$ and $\alpha^{\eta_o\lambda_o}_{max}$, where weights are given by the relative positions of $A^{\eta_o\lambda_o}_{min}$, $A^{\eta_o\lambda_o}_{max}$ and A_o:

$$\boxed{\begin{cases} \varphi_o = \varphi^{\eta_o\lambda_o} \;, \\ \alpha_o = \alpha^{\eta_o\lambda_o}_{min} + \dfrac{A_o - A^{\eta_o\lambda_o}_{min}}{A^{\eta_o\lambda_o}_{max} - A^{\eta_o\lambda_o}_{min}} \left(\alpha^{\eta_o\lambda_o}_{max} - \alpha^{\eta_o\lambda_o}_{min}\right) \end{cases}}$$

5 Extension to heterogeneous resources

The previous work concerns an optimization of resources or sensors toward the best detection of a target. However, an implicit hypothesis is made on the sensors in use, namely they are of same type. Nevertheless, the present work may be extended to multi-type optimization problems within the same general formalism. More precisely, each type of resource will be represented by a type index $\rho \in \{1, \dots, r\}$. Then, for each type of resource ρ, are defined a specific non detection probability function p^ρ and an associated resource allocation function φ^ρ. The total amount of resources of type ρ is also denoted ϕ^ρ_o. A proper constraint holds for each resource type:

$$\forall \rho, \int_E \varphi^\rho(x)\, dx \leq \phi^\rho_o \;.$$

At last, the non detection probabilities for two differents types are supposed independant. Thus, the global non detection probability (i.e. the game value) is given by:

$$P_{nd}(\alpha, \varphi) = \int_E \alpha(x) \prod_{\rho=1}^{r} p^\rho_x(\varphi^\rho(x))\, dx \;.$$

As already seen, equality constraints may be used instead of inequality constraints. Some properties are again assumed about the non detection functions p^ρ_x. However, the only hypothesis $p^{\rho\,\prime}_x < 0$ and $p^{\rho\,\prime\prime}_x > 0$ are not sufficient, since they do not guarantee the convexity on evaluation function. However, a natural property on resource efficiency yields good behavior of the non detection function. In fact, a natural assumption is that the power of resources is even or decrease with concentration (refer to Appendix C). Formally, this signifies that the non detection probability is of form:

$$p_x^\rho(\varphi) = \exp(-w_x^\rho(\varphi)) \ ,$$

where w_x^ρ is a positive and **concave** function of the local resource amount φ. The game problem may be written as:

Find:

$$\alpha_o = \arg\max_\alpha \min_\varphi \int_E \alpha(x) \prod_{\rho=1}^{r} p_x^\rho(\varphi^\rho(x))\, dx$$

and

$$\varphi_o = \arg\min_\varphi \max_\alpha \int_E \alpha(x) \prod_{\rho=1}^{r} p_x^\rho(\varphi^\rho(x))\, dx \ ,$$

under constraints:

$$\forall x \in E,\ \alpha_1(x) \leq \alpha(x) \leq \alpha_2(x) \ , \text{ and } \int_E \alpha(x)\, dx = A_o \ ,$$

$$\forall \rho,\ \forall x \in E,\ \varphi_1^\rho(x) \leq \varphi^\rho(x) \leq \varphi_2^\rho(x) \ , \text{ and } \forall \rho,\ \int_E \varphi^\rho(x)\, dx = \phi_o^\rho \ .$$

Quite similarly to the previous case, similar optimality conditions are straightforwardly obtained for this enriched formalism:

$$\begin{cases} \alpha_1(x) < \alpha_o(x) < \alpha_2(x) \Longrightarrow \prod_{\rho=1}^{\rho=r} p_x^\rho(\varphi_o^\rho(x)) = \lambda \ , \\ \prod_{\rho=1}^{\rho=r} p_x^\rho(\varphi_o^\rho(x)) > \lambda \Longrightarrow \alpha_o(x) > \alpha_1(x) \ , \\ \prod_{\rho=1}^{\rho=r} p_x^\rho(\varphi_o^\rho(x)) < \lambda \Longrightarrow \alpha_o(x) < \alpha_2(x) \ . \end{cases}$$

and

$$\begin{cases} \varphi_1^\varrho(x) < \varphi_o^\varrho(x) < \varphi_2^\varrho(x) \Longrightarrow \alpha_o(x) \left(\prod_{1\le\rho\le r}^{\rho\neq\varrho} p_x^\rho(\varphi_o^\rho(x)) \right) p_x^{\varrho\prime}(\varphi_o^\varrho(x)) = \eta^\varrho \ , \\ \alpha_o(x) > \dfrac{\eta^\varrho}{\left(\prod_{1\le\rho\le r}^{\rho\neq\varrho} p_x^\rho(\varphi_o^\rho(x)) \right) p_x^{\varrho\prime}(\varphi_1^\varrho(x))} \Longrightarrow \varphi_o^\varrho(x) > \varphi_1^\varrho(x) \ , \\ \alpha_o(x) < \dfrac{\eta^\varrho}{\left(\prod_{1\le\rho\le r}^{\rho\neq\varrho} p_x^\rho(\varphi_o^\rho(x)) \right) p_x^{\varrho\prime}(\varphi_2^\varrho(x))} \Longrightarrow \varphi_o^\varrho(x) < \varphi_2^\varrho(x) \ , \end{cases}$$

for all $\varrho \in \{1 \dots r\}$. But these equations are uneasy to invert. In this section, an iterative optimization process will be proposed. For this process, a step will consist in a min-max optimization of target strategy within the strategy of one type of resource (alone). For the concerned step, the strategy of the other types of resource are maintained to the value of the previous step. This method can be implemented simply as a direct extension of the main algorithm described in this paper. However, the process constructs a searcher strategy more and more efficient, increasing continuously the worse game value. The final process steps yield both an optimal multi-type searcher strategy and an optimal target strategy.

Process step definition: Number p will represent a processing step index. Values of p are non negative integers. For a given value p, the unique integer $\rho \in \{1, \dots, r\}$, such that $p - \rho$ is a r-multiple, is denoted $p[r]$. Functional sequence $\left(\alpha_{(p)}\right)_p$ and $\left(\varphi_{(p)}^\rho\right)_{\rho,p}$ and game evaluation sequence $(\mathcal{V}_{(p)})_p$ are defined as follows:

$$\begin{cases} \alpha_{(0)} \text{ undefined} \,;\ \forall \rho,\ \varphi^{\rho}_{(0)} = \dfrac{\varphi^{\rho}_1 + \varphi^{\rho}_2}{2} \,;\ \mathcal{V}_{(0)} \text{ undefined} \,, \\ \alpha_{(p+1)} \in \arg\max_{\alpha} \min_{\varphi^{p[r]}} \displaystyle\int_E \alpha(x) \left(\prod_{1\leq\rho\leq r}^{\rho\neq p[r]} p^{\rho}_x(\varphi^{\rho}_{(p)}(x)) \right) p^{p[r]}_x(\varphi^{p[r]}(x))\, dx \\ \varphi^{p[r]}_{(p+1)} \in \arg\min_{\varphi^{p[r]}} \max_{\alpha} \displaystyle\int_E \alpha(x) \left(\prod_{1\leq\rho\leq r}^{\rho\neq p[r]} p^{\rho}_x(\varphi^{\rho}_{(p)}(x)) \right) p^{p[r]}_x(\varphi^{p[r]}(x))\, dx \\ \forall \rho \neq p[r],\ \varphi^{\rho}_{(p+1)} = \varphi^{\rho}_{(p)} \,. \\ \mathcal{V}_{(p+1)} = \displaystyle\int_E \alpha_{(p+1)}(x) \left(\prod_{\rho=1}^{\rho=r} p^{\rho}_x(\varphi^{\rho}_{(p+1)}(x)) \right) dx \end{cases}$$

This sequence is obviously computable by means of the algorithm, since it corresponds to a sequence of one-type game optimization. However, this sequence has a good increaseness property on $\mathcal{V}$. As a direct consequence of definition, all choice of $\varphi^{p[r]}$ verifies the inequality:

$$\mathcal{V}_{(p+1)} \leq \max_{\alpha} \int_E \alpha(x) \left(\prod_{1\leq\rho\leq r}^{\rho\neq p[r]} p^{\rho}_x(\varphi^{\rho}_{(p)}(x)) \right) p^{p[r]}_x(\varphi^{p[r]}(x))\, dx \,.$$

In particular, setting $\varphi^{p[r]} = \varphi^{p[r]}_{(p)}$ permits to recover $\mathcal{V}_{(p)}$. Hence the property:

$$\forall p \geq 1,\ \mathcal{V}_{(p)} \geq \mathcal{V}_{(p+1)}$$

Since $(\mathcal{V}_{(p)})_p$ is a sequence of non negative number, it then admits a limit denoted $\mathcal{V}_{(\infty)}$. We are going to show that the last strategies $\alpha_{(p)}$ and $\varphi^{\rho}_{(p)}$, for $p \sim \infty$, verify the optimality equations. Let us define:

$$\mathcal{V}(\varphi) = \max_{\alpha} \int_E \alpha(x) \left(\prod_{\rho=1}^{\rho=r} p^{\rho}_x(\varphi^{\rho}(x)) \right) dx \,,$$

then the study of the algorithm behavior relies on the following lemmas.

Lemma 3 *The function $\mathcal{V}$ is continue.*

The $\mathcal{V}$ function is a max of continuous functions and thus is itself continuous.

Lemma 4 *The function $\mathcal{V}$ is convex.*

proof: Let φ_a and φ_b be two resource sharing functions. Let $\theta \in [0,1]$. From the concavity of w_x^ρ, property:

$$w_x^\rho(\theta\varphi_a^\rho(x) + (1-\theta)\varphi_b^\rho(x)) \geq \theta w_x^\rho(\varphi_a^\rho(x)) + (1-\theta)w_x^\rho(\varphi_b^\rho(x))$$

holds true for each $\rho \in \{1, \dots, r\}$. It follows:

$$\sum_{\rho=1}^{\rho=r} w_x^\rho(\theta\varphi_a^\rho(x)+(1-\theta)\varphi_b^\rho(x)) \geq \theta \sum_{\rho=1}^{\rho=r} w_x^\rho(\varphi_a^\rho(x))+(1-\theta)\sum_{\rho=1}^{\rho=r} w_x^\rho(\varphi_b^\rho(x)) \, .$$

Now, exp is a convex and increasing function and thus:

$$\prod_{\rho=1}^{\rho=r} p_x^\rho(\theta\varphi_a^\rho(x) + (1-\theta)\varphi_b^\rho(x)) \leq \theta \prod_{\rho=1}^{\rho=r} p_x^\rho(\varphi_a^\rho(x)) + (1-\theta)\prod_{\rho=1}^{\rho=r} p_x^\rho(\varphi_b^\rho(x)) \, .$$

It follows:

$$\int_E \alpha(x) \left(\prod_{\rho=1}^{\rho=r} p_x^\rho(\theta\varphi_a^\rho(x) + (1-\theta)\varphi_b^\rho(x))\right) dx \leq \theta\mathcal{V}(\varphi_a^\rho(x))+(1-\theta)\mathcal{V}(\varphi_b^\rho(x)) \, .$$

Ending the proof.

□□□

Lemma 5 *Let $\varrho \in \{1, \dots, r\}$ and φ^ρ be fixed for $\rho \neq \varrho$. Assume $\alpha_1 > 0$. Then, the function $\varphi^\varrho \mapsto \mathcal{V}(\varphi)$ is strictly convex. In particular, there is a unique minimizer for the function $\varphi^\varrho \mapsto \mathcal{V}(\varphi)$.*

proof: Functions φ^ρ are given for $\rho \neq \varrho$. Let φ_a^ϱ and φ_b^ϱ be two distinct (i.e. the difference between φ_a^ϱ and φ_b^ϱ has a non-zero measure) resource sharing functions for resource ϱ. Let $\theta \in]0,1[$. From the concavity of w_x^ϱ and since exp is a convex function, we have:

$$p_x^\varrho\left(\theta\varphi_a^\varrho(x)+(1-\theta)\varphi_b^\varrho(x)\right)\prod_{\rho\neq\varrho}p_x^\rho\left(\varphi^\rho(x)\right)\le\theta p_x^\varrho\left(\varphi_a^\varrho(x)\right)\prod_{\rho\neq\varrho}p_x^\rho\left(\varphi^\rho(x)\right)+ (1-\theta)p_x^\varrho\left(\varphi_b^\varrho(x)\right)\prod_{\rho\neq\varrho}p_x^\rho\left(\varphi^\rho(x)\right)\ .$$

On the other hand, the function exp is also strictly convex and some results may be refined:

$$\begin{cases} |\mathfrak{m}-\mathfrak{n}|\ge\epsilon\Rightarrow\exp(\theta\mathfrak{m}+(1-\theta)\mathfrak{n})\le\dfrac{\exp(\theta\epsilon)}{1+\theta(\exp\epsilon-1)}(\theta\exp\mathfrak{m}+(1-\theta)\exp\mathfrak{n})\ , \\ \forall\epsilon>0,\ \dfrac{\exp(\theta\epsilon)}{1+\theta(\exp\epsilon-1)}<1\ . \end{cases}$$

Now, w_x^ϱ is increasing. Since φ_a^ϱ and φ_b^ϱ are measurably distinct, there is then a non negligible set ε such that:

$$x\in\varepsilon\Rightarrow\left|w_x^\varrho\left(\varphi_a^\varrho(x)\right)-w_x^\varrho\left(\varphi_b^\varrho(x)\right)\right|\ge\epsilon\ .$$

It follows:

$$\forall x\in\varepsilon,\ p_x^\varrho\left(\theta\varphi_a^\varrho(x)+(1-\theta)\varphi_b^\varrho(x)\right)\prod_{\rho\neq\varrho}p_x^\rho\left(\varphi^\rho(x)\right)\le\frac{\exp(\theta\epsilon)}{1+\theta(\exp\epsilon-1)}\times \left(\theta p_x^\varrho\left(\varphi_a^\varrho(x)\right)\prod_{\rho\neq\varrho}p_x^\rho\left(\varphi^\rho(x)\right)+(1-\theta)p_x^\varrho\left(\varphi_b^\varrho(x)\right)\prod_{\rho\neq\varrho}p_x^\rho\left(\varphi^\rho(x)\right)\right)\ .$$

Integrating on ε and $E \setminus \varepsilon$, we thus have:

$$\int_E \alpha(x) p_x^{\varrho}\left(\theta\varphi_a^{\varrho}(x) + (1-\theta)\varphi_b^{\varrho}(x)\right) \prod_{\rho\neq\varrho} p_x^{\rho}\left(\varphi^{\rho}(x)\right)\, dx \leq$$
$$\theta \int_E \alpha(x) p_x^{\varrho}\left(\varphi_a^{\varrho}(x)\right) \prod_{\rho\neq\varrho} p_x^{\rho}\left(\varphi^{\rho}(x)\right)\, dx +$$
$$(1-\theta)\int_E \alpha(x) p_x^{\varrho}\left(\varphi_b^{\varrho}(x)\right) \prod_{\rho\neq\varrho} p_x^{\rho}\left(\varphi^{\rho}(x)\right)\, dx +$$
$$\left(\frac{\exp(\theta\epsilon)}{1+\theta(\exp\epsilon - 1)} - 1\right) \int_\varepsilon \alpha(x) \Big(\theta p_x^{\varrho}\left(\varphi_a^{\varrho}(x)\right) +$$
$$(1-\theta) p_x^{\varrho}\left(\varphi_b^{\varrho}(x)\right)\Big) \prod_{\rho\neq\varrho} p_x^{\rho}\left(\varphi^{\rho}(x)\right)\, dx\ .$$

Since $\left(\frac{\exp(\theta\epsilon)}{1+\theta(\exp\epsilon-1)} - 1\right) < 0$, it follows:

$$\mathcal{V}(\varphi^{\rho}|_{\rho<\varrho}, \theta\varphi_a^{\varrho} + (1-\theta)\varphi_b^{\varrho}, \varphi^{\rho}|_{\rho>\varrho}) \leq$$
$$\theta\mathcal{V}(\varphi^{\rho}|_{\rho<\varrho}, \varphi_a^{\varrho}, \varphi^{\rho}|_{\rho>\varrho}) + (1-\theta)\mathcal{V}(\varphi^{\rho}|_{\rho<\varrho}, \varphi_b^{\varrho}, \varphi^{\rho}|_{\rho>\varrho}) +$$
$$\left(\frac{\exp(\theta\epsilon)}{1+\theta(\exp\epsilon - 1)} - 1\right) \int_\varepsilon \alpha_1(x) \Big(\theta p_x^{\varrho}\left(\varphi_a^{\varrho}(x)\right) +$$
$$(1-\theta) p_x^{\varrho}\left(\varphi_b^{\varrho}(x)\right)\Big) \prod_{\rho\neq\varrho} p_x^{\rho}\left(\varphi^{\rho}(x)\right)\, dx\ .$$

Now, ε is non negligible and $\alpha_1 > 0$, so that:

$$\mathcal{V}(\varphi^{\rho}|_{\rho<\varrho}, \theta\varphi_a^{\varrho} + (1-\theta)\varphi_b^{\varrho}, \varphi^{\rho}|_{\rho>\varrho}) <$$
$$\theta\mathcal{V}(\varphi^{\rho}|_{\rho<\varrho}, \varphi_a^{\varrho}, \varphi^{\rho}|_{\rho>\varrho}) + (1-\theta)\mathcal{V}(\varphi^{\rho}|_{\rho<\varrho}, \varphi_b^{\varrho}, \varphi^{\rho}|_{\rho>\varrho})\ ,$$

which achieves the proof of lemma 5.

□□□

Now, let $\varphi_{(\infty)} \in \lim_{p\to\infty} \varphi_{(p)}$ a value of adherence of the sequence $(\varphi_{(p)})_p$. Then, we can suppose $\varphi_{(\infty)} \in \lim_{p\to\infty} \varphi_{(rp+1)}$, without loss of generality. Then, let

$(u_n)_n$ be increasing integer sequence, such that $\lim_{n\to\infty} \varphi_{(ru_n+1)} = \varphi_{(\infty)}$. By hypothesis:

$$\varphi^1_{(ru_n+1)} = \arg\min_{\varphi^1} \mathcal{V}\left(\varphi^1, \varphi^\rho_{(ru_n+1)}|_{\rho>1}\right) \ .$$

Thus:

$$\forall \varphi^1,\ \mathcal{V}\left(\varphi^1, \varphi^\rho_{(ru_n+1)}|_{\rho>1}\right) \geq \mathcal{V}\left(\varphi_{(ru_n+1)}\right) \ .$$

Finally, the continuity of $\mathcal{V}$ yields $\forall \varphi^1,\ \mathcal{V}\left(\varphi^1, \varphi^\rho_{(\infty)}|_{\rho>1}\right) \geq \mathcal{V}\left(\varphi_{(\infty)}\right)$ and:

$$\varphi^1_{(\infty)} = \arg\min_{\varphi^1} \mathcal{V}\left(\varphi^1, \varphi^\rho_{(\infty)}|_{\rho>1}\right) \ .$$

On the other hand, also holds from definition:

$$\forall \varphi^2,\ \mathcal{V}\left(\varphi^1_{(ru_n+2)}, \varphi^2, \varphi^\rho_{(ru_n+2)}|_{\rho>2}\right) \geq \mathcal{V}\left(\varphi_{(ru_n+2)}\right) \ . \tag{24}$$

Now, $\varphi^\rho_{(ru_n+2)} = \varphi^\rho_{(ru_n+1)}$ for $\rho \neq 2$ and $\lim_{n\to\infty} \mathcal{V}\left(\varphi_{(ru_n+2)}\right) = \mathcal{V}_{(\infty)}$, and since $\mathcal{V}$ is a continuous function and $\mathcal{V}_{(\infty)} = \mathcal{V}(\varphi_{(\infty)})$, we have:

$$\forall \varphi^2,\ \mathcal{V}\left(\varphi^1_{(\infty)}, \varphi^2, \varphi^\rho_{(\infty)}|_{\rho>2}\right) \geq \mathcal{V}\left(\varphi_{(\infty)}\right) \ ,$$

so that:

$$\varphi^2_{(\infty)} = \arg\min_{\varphi^2} \mathcal{V}\left(\varphi^1_{(\infty)}, \varphi^2, \varphi^\rho_{(\infty)}|_{\rho>2}\right) \ .$$

So, the result of this step is quite similar to the previous one. However, it is obtained in a different manner and the process may not be continued for the other steps. Now, let $\varphi^2_{(\aleph)} \in \lim_{n\to\infty} \varphi^2_{(ru_n+2)}$. From equation (24) and since $\forall \rho \neq 2,\ \varphi^\rho_{(ru_n+2)} = \varphi^\rho_{(ru_n+1)}$, it is easily shown that:

$$\forall \varphi^2,\ \mathcal{V}\left(\varphi^1_{(\infty)}, \varphi^2, \varphi^\rho_{(\infty)}|_{\rho>2}\right) \geq \mathcal{V}\left(\varphi^1_{(\infty)}, \varphi^2_{(\aleph)}, \varphi^\rho_{(\infty)}|_{\rho>2}\right) \ .$$

Because of the strict convexity of lemma 5, the minimizer $\varphi^2_{(\infty)}$ is unique and thus, $\varphi^2_{(\infty)} = \varphi^2_{(\aleph)}$. We have just shown that $\lim_{n\to\infty} \varphi_{(ru_n+2)} = \varphi_{(\infty)}$. The process may be continued and the following holds:

$$\forall k \geq 0,\ \lim_{n\to\infty} \varphi_{(ru_n+k)} = \varphi_{(\infty)}$$

and

$$\forall \varrho \in [\![1, r]\!]\,,\ \varphi^{\varrho}_{(\infty)} = \arg\min_{\varphi^{\varrho}} \mathcal{V}\left(\varphi^{\rho}_{(\infty)}|_{\rho>\varrho}, \varphi^{\varrho}, \varphi^{\rho}_{(\infty)}|_{\rho>\varrho}\right)\ .$$

Considering min-max strategies as saddle-points, the definition of $\alpha_{(p+1)}$ may be changed into:

$$\alpha_{(p+1)} \in \arg\max_{\alpha} \int_E \alpha(x) \prod_{1\leq\rho\leq r} p^{\rho}_x(\varphi^{\rho}_{(p+1)}(x))\, dx$$

So, making hypothesis $\lim_{n\to\infty} \alpha_{(u_n)} = \alpha_{(\infty)}$ and $\lim_{n\to\infty} \varphi_{(u_n)} = \varphi_{(\infty)}$ yields thanks to continuousness:

$$\alpha_{(\infty)} \in \arg\max_{\alpha} \int_E \alpha(x) \prod_{1\leq\rho\leq r} p^{\rho}_x(\varphi^{\rho}_{(\infty)}(x))\, dx$$

Finally:

$$\boxed{\forall \begin{pmatrix}\alpha_{(\infty)}\\ \varphi_{(\infty)}\end{pmatrix} \in \lim_{p\to\infty} \begin{pmatrix}\alpha_{(p)}\\ \varphi_{(p)}\end{pmatrix}\,,\ \begin{cases}\forall \varrho\,,\ \varphi^{\varrho}_{(\infty)} = \arg\min_{\varphi^{\varrho}} \mathcal{V}\left(\varphi^{\rho}_{(\infty)}|_{\rho>\varrho}, \varphi^{\varrho}, \varphi^{\rho}_{(\infty)}|_{\rho>\varrho}\right)\\ \alpha_{(\infty)} \in \arg\max_{\alpha} \int_E \alpha(x) \prod_{1\leq\rho\leq r} p^{\rho}_x(\varphi^{\rho}_{(\infty)}(x))\, dx\end{cases}} \tag{25}$$

This property indicates that the results of convergence satisfy fortunately the optimality equations.

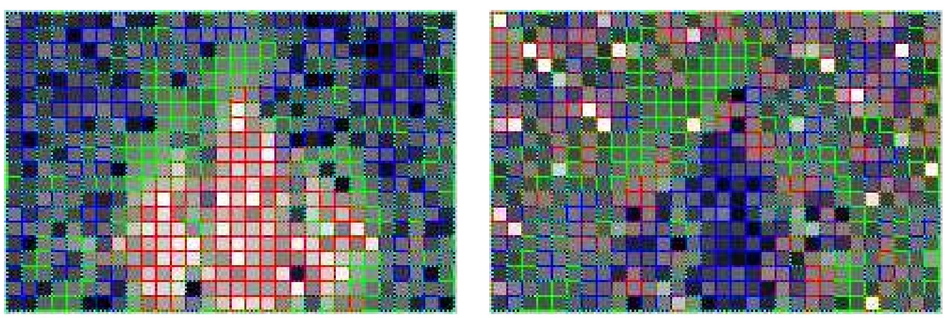
6 Results

6.1 One-type game

In this section, we present an exemple for one-type game computed by the basic algorithm. The search space E is a set of 30×20 cells. Values $A_o = 1$ and $\phi_o = 30$ are used. The local bounds α_1, α_2, φ_1 and φ_2 are represented in first frames of figure 4. In the figures, dark cells are representing low values, while bright cells represent high values. The conditional probability, p, is of exponential form $p_x(\varphi) = \exp(-\omega_x\varphi)$. The visibility parameter ω_x is weak for poor detection and high for good detection. The parameter ω is represented by last frame of figure 4. The functions α_o and φ_o obtained after convergence are represented in figures 5. Again, low values correspond to dark cells whereas bright cells represent high values. Moreover, the color of the cell contours indicate if bounds are reached or not. Blue contour on cell x means $\varphi_o(x) = \varphi_1(x)$ or $\alpha_o(x) = \alpha_1(x)$. Green contour on cell x signifies $\varphi_1(x) < \varphi_o(x) < \varphi_2(x)$ or $\alpha_1(x) < \alpha_o(x) < \alpha_2(x)$. Red contour on cell x corresponds to $\varphi_o(x) = \varphi_2(x)$ or $\alpha_o(x) = \alpha_2(x)$.

α_1 α_2 φ_1 φ_2 Parameter ω

Figure 4: Game description.

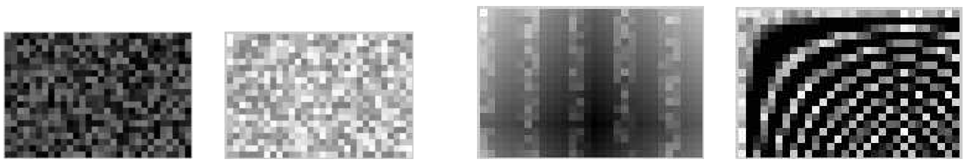


α_o φ_o

Figure 5: Strategies.

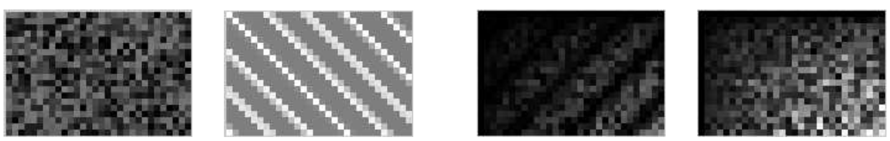
6.2 Multi-type game

In this section, we present an exemple for a two-type game computed by the iterated algorithm. The search space E is a set of 30×20 cells. Two types of resources, a and b, are used. The global constraints are given by $A_o = 1$, $\phi^a_o = 30$ and $\phi^b_o = 20$. The local bounds α_1 and α_2 are represented in the two first frames of figure 6. The local bounds φ^a_1, φ^a_2, φ^b_1 and φ^b_2 are represented in figure 7. The conditional probabilities, p^a and p^b, are still of exponential form, $p^a_x(\varphi^a) = \exp(-\omega^a_x \varphi^a)$ and $p^b_x(\varphi^b) = \exp(-\omega^b_x \varphi^b)$. The parameters ω^a and ω^b are represented by the two last frames of figure 6. The functions $\alpha_{(\infty)}$, $\varphi^a_{(\infty)}$ and $\varphi^b_{(\infty)}$ obtained after convergence are represented in figures 8.



α_1 α_2 Parameter ω^a Parameter ω^b

Figure 6: Game description.



φ^a_1 φ^a_2 φ^b_1 φ^b_2

Figure 7: Game description.

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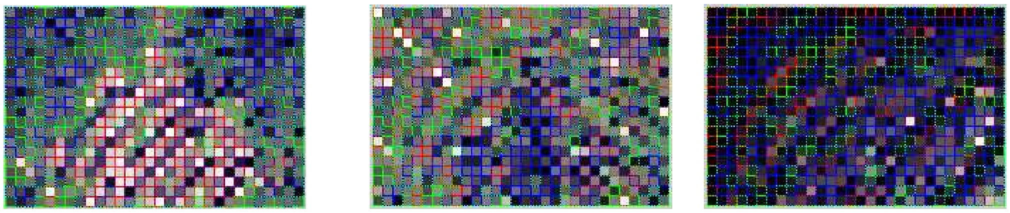

$\alpha_{(\infty)}$ $\varphi^a_{(\infty)}$ $\varphi^b_{(\infty)}$

Figure 8: Strategies.

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A Computing $\alpha^{\eta\lambda}_{min}$, $\alpha^{\eta\lambda}_{max}$ and $\varphi^{\eta\lambda}$.

The purpose of this section is to give a systematic method for computing $\alpha^{\eta\lambda}_{min}$, $\alpha^{\eta\lambda}_{max}$ and $\varphi^{\eta\lambda}$, when variables η and λ are given (proofs are left to the reader). Theoretically, this problem is very simple, but implementation is uneasy because lot of cases have to be checked. So, let η, λ and $x \in E$ be given. Five main cases are considered.

Case I: $p_x^{-1}(\lambda) < \varphi_1(x)$.
Values of $\alpha^{\eta\lambda}_{min}(x)$ and $\alpha^{\eta\lambda}_{max}(x)$ are directly stated:

$$\alpha^{\eta\lambda}_{min}(x) = \alpha^{\eta\lambda}_{max}(x) = \alpha_1(x) \ .$$

For defining $\varphi^{\eta\lambda}(x)$, three subcases are considered:

case a: $\alpha_1(x) < \frac{\eta}{p'_x(\varphi_1(x))}$. Then $\varphi^{\eta\lambda}(x) = \varphi_1(x)$.

case b: $\frac{\eta}{p'_x(\varphi_1(x))} \leq \alpha_1(x) \leq \frac{\eta}{p'_x(\varphi_2(x))}$. Then $\varphi^{\eta\lambda}(x) = {p'_x}^{-1}\left(\frac{\eta}{\alpha_1(x)}\right)$.

case c: $\alpha_1(x) > \frac{\eta}{p'_x(\varphi_2(x))}$. Then $\varphi^{\eta\lambda}(x) = \varphi_2(x)$.

Case II: $p_x^{-1}(\lambda) = \varphi_1(x)$.
Four subcases are considered here:

case a: $\frac{\eta}{p'_x(\varphi_1(x))} > \alpha_2(x)$. Then $\varphi^{\eta\lambda}(x) = \varphi_1(x)$, $\alpha^{\eta\lambda}_{min}(x) = \alpha_1(x)$ and $\alpha^{\eta\lambda}_{max}(x) = \alpha_2(x)$.

case b: $\alpha_1(x) \leq \frac{\eta}{p'_x(\varphi_1(x))} \leq \alpha_2(x)$. Then $\varphi^{\eta\lambda}(x) = \varphi_1(x)$, $\alpha^{\eta\lambda}_{min}(x) = \alpha_1(x)$ and $\alpha^{\eta\lambda}_{max}(x) = \frac{\eta}{p'_x(\varphi_1(x))}$.

case c: $\frac{\eta}{p'_x(\varphi_1(x))} < \alpha_1(x) \leq \frac{\eta}{p'_x(\varphi_2(x))}$. Then $\alpha^{\eta\lambda}_{min}(x) = \alpha^{\eta\lambda}_{max}(x) = \alpha_1(x)$ and $\varphi^{\eta\lambda}(x) = {p'_x}^{-1}\left(\frac{\eta}{\alpha_1(x)}\right)$.

case d: $\frac{\eta}{p'_x(\varphi_2(x))} < \alpha_1(x)$. Then $\alpha^{\eta\lambda}_{min}(x) = \alpha^{\eta\lambda}_{max}(x) = \alpha_1(x)$ and $\varphi^{\eta\lambda}(x) = \varphi_2(x)$.

Case III: $\varphi_1(x) < p_x^{-1}(\lambda) < \varphi_2(x)$.
First, define the numbers φ_L and φ_R by:

$$\left\{\begin{array}{l} \alpha_1(x) < \frac{\eta}{p'_x(\varphi_1(x))} \Rightarrow \varphi_L = \varphi_1(x)\ , \\ \frac{\eta}{p'_x(\varphi_1(x))} \leq \alpha_1(x) \leq \frac{\eta}{p'_x(\varphi_2(x))} \Rightarrow \varphi_L = {p'_x}^{-1}\left(\frac{\eta}{\alpha_1(x)}\right)\ , \\ \alpha_1(x) > \frac{\eta}{p'_x(\varphi_2(x))} \Rightarrow \varphi_L = \varphi_2(x)\ , \end{array}\right.$$

and

$$\left\{\begin{array}{l} \alpha_2(x) > \frac{\eta}{p'_x(\varphi_2(x))} \Rightarrow \varphi_R = \varphi_2(x)\ , \\ \frac{\eta}{p'_x(\varphi_1(x))} \leq \alpha_2(x) \leq \frac{\eta}{p'_x(\varphi_2(x))} \Rightarrow \varphi_R = {p'_x}^{-1}\left(\frac{\eta}{\alpha_2(x)}\right)\ , \\ \alpha_2(x) < \frac{\eta}{p'_x(\varphi_1(x))} \Rightarrow \varphi_R = \varphi_1(x)\ . \end{array}\right.$$

Then, three cases are considered:

case a: $\varphi_L > p_x^{-1}(\lambda)$. Then $\alpha_{min}^{\eta\lambda}(x) = \alpha_{max}^{\eta\lambda}(x) = \alpha_1(x)$ and $\varphi^{\eta\lambda}(x) = \varphi_L$.

case b: $\varphi_L \leq p_x^{-1}(\lambda) \leq \varphi_R$. Then $\alpha_{min}^{\eta\lambda}(x) = \alpha_{max}^{\eta\lambda}(x) = \frac{\eta}{p'_x\left(p_x^{-1}(\lambda)\right)}$ and $\varphi^{\eta\lambda}(x) = p_x^{-1}(\lambda)$.

case c: $\varphi_R < p_x^{-1}(\lambda)$. Then $\alpha_{min}^{\eta\lambda}(x) = \alpha_{max}^{\eta\lambda}(x) = \alpha_2(x)$ and $\varphi^{\eta\lambda}(x) = \varphi_R$.

Case IV: $p_x^{-1}(\lambda) = \varphi_2(x)$.
Four subcases are considered here:

case a: $\frac{\eta}{p'_x(\varphi_2(x))} < \alpha_1(x)$. Then $\varphi^{\eta\lambda}(x) = \varphi_2(x)$, $\alpha_{min}^{\eta\lambda}(x) = \alpha_1(x)$ and $\alpha_{max}^{\eta\lambda}(x) = \alpha_2(x)$.

case b: $\alpha_1(x) \leq \frac{\eta}{p'_x(\varphi_2(x))} \leq \alpha_2(x)$. Then $\varphi^{\eta\lambda}(x) = \varphi_2(x)$, $\alpha_{min}^{\eta\lambda}(x) = \frac{\eta}{p'_x(\varphi_2(x))}$ and
$\alpha_{max}^{\eta\lambda}(x) = \alpha_2(x)$.

case c: $\frac{\eta}{p'_x(\varphi_1(x))} \leq \alpha_2(x) < \frac{\eta}{p'_x(\varphi_2(x))}$. Then $\alpha_{min}^{\eta\lambda}(x) = \alpha_{max}^{\eta\lambda}(x) = \alpha_2(x)$ and $\varphi^{\eta\lambda}(x) = {p'_x}^{-1}\left(\frac{\eta}{\alpha_2(x)}\right)$.

case d: $\frac{\eta}{p'_x(\varphi_1(x))} > \alpha_2(x)$. Then $\alpha_{min}^{\eta\lambda}(x) = \alpha_{max}^{\eta\lambda}(x) = \alpha_2(x)$ and $\varphi^{\eta\lambda}(x) = \varphi_1(x)$.

Case V: $p_x^{-1}(\lambda) > \varphi_2(x)$.
Values of $\alpha_{min}^{\eta\lambda}(x)$ and $\alpha_{max}^{\eta\lambda}(x)$ are directly stated:

$$\alpha_{min}^{\eta\lambda}(x) = \alpha_{max}^{\eta\lambda}(x) = \alpha_2(x)\ .$$

For defining $\varphi^{\eta\lambda}(x)$, three subcases are considered:

case a: $\alpha_2(x) > \frac{\eta}{p'_x(\varphi_2(x))}$. Then $\varphi^{\eta\lambda}(x) = \varphi_2(x)$.

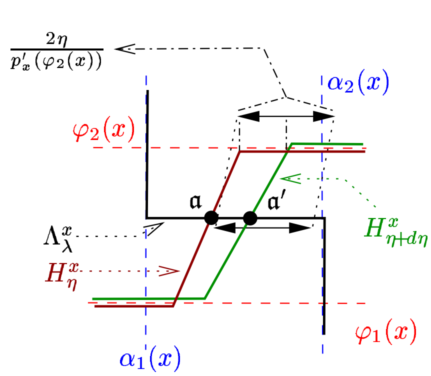
case b: $\frac{\eta}{p'_x(\varphi_1(x))} \leq \alpha_2(x) \leq \frac{\eta}{p'_x(\varphi_2(x))}$. Then $\varphi^{\eta\lambda}(x) = {p'_x}^{-1}\left(\frac{\eta}{\alpha_2(x)}\right)$.

case c: $\alpha_2(x) < \frac{\eta}{p'_x(\varphi_1(x))}$. Then $\varphi^{\eta\lambda}(x) = \varphi_1(x)$.

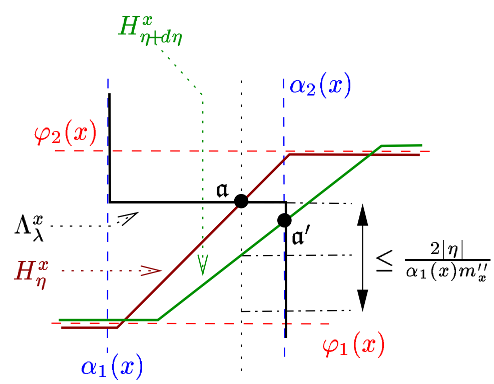
B Proof of lemma 2

Only the first implication is proven, since the other case is similar. Now, let $d\eta < 0$ and $(\mathfrak{a}, \mathfrak{f}) \in H^x_\eta \cap \Lambda^x_\lambda$. From lemma 1 exists $(\mathfrak{a} + \delta\mathfrak{a}, \mathfrak{f} + \delta\mathfrak{f})$, with $\delta\mathfrak{a} \geq 0$ and $\delta\mathfrak{f} \leq 0$, such that $(\mathfrak{a} + \delta\mathfrak{a}, \mathfrak{f} + \delta\mathfrak{f}) \in H^x_{\eta+d\eta} \cap \Lambda^x_\lambda$. Property (26) and (27) are now proven in order to specify variations $\delta\mathfrak{a}$ and $\delta\mathfrak{f}$ (refer to figure 9 for some illustration).

$$\left.\begin{array}{r} \mathfrak{a}' \geq \mathfrak{a} + \dfrac{2d\eta}{p'_x(\varphi_2(x))} \\ \mathfrak{f}' \leq \mathfrak{f}\ ,\ \varphi_1(x) < \mathfrak{f}\ ,\ \mathfrak{f}' < \varphi_2(x) \end{array}\right\} \Rightarrow (\mathfrak{a}', \mathfrak{f}') \notin H^x_{\eta+d\eta}\ . \qquad (26)$$



Eq. (26)



Eq. (27)

Figure 9: Proof of implications

proof: We now make the hypotheses of the implication (26). By definition of H^x_η, the hypothesis $\varphi_1(x) < \mathfrak{f}$ yields $\mathfrak{a} \geq \frac{\eta}{p'_x(\mathfrak{f})}$. Thus, $\mathfrak{a}' \geq \frac{\eta}{p'_x(\mathfrak{f})} + \frac{2d\eta}{p'_x(\varphi_2(x))}$ holds true. Since $p' < 0$, property $\mathfrak{a}' p'_x(\mathfrak{f}') \leq \left(\frac{\eta}{p'_x(\mathfrak{f})} + \frac{2d\eta}{p'_x(\varphi_2(x))}\right) p'_x(\mathfrak{f})$ is obtained. Now,

$p'_x(\varphi_2(x)) \geq p'_x(\mathfrak{f})$ and $\mathfrak{a}'p'_x(\mathfrak{f}') \leq \eta + 2d\eta < \eta + d\eta$. Property $\mathfrak{a}'p'_x(\mathfrak{f}') < \eta + d\eta$ added to the hypothesis $\mathfrak{f}' < \varphi_2(x)$ yields $(\mathfrak{a}',\mathfrak{f}') \notin H^x_{\eta+d\eta}$, thanks to definition of $H^x_{\eta+d\eta}$.

□□□

$$\left.\begin{array}{r}\mathfrak{a}' \geq \mathfrak{a}\\ \mathfrak{f}' \leq \mathfrak{f} + \dfrac{2d\eta}{\alpha_1(x)m''_x}\end{array}\right\} \Rightarrow (\mathfrak{a}',\mathfrak{f}') \notin H^x_{\eta+d\eta}\ . \tag{27}$$

proof: Make both the hypotheses $\mathfrak{a}' \geq \mathfrak{a}$ and $\mathfrak{f}' \leq \mathfrak{f} + \frac{2d\eta}{\alpha_1(x)m''_x}$. Since p'_x is increasing, the property $\mathfrak{a}'p'_x(\mathfrak{f}') \leq \mathfrak{a}p'_x\left(\mathfrak{f} + \frac{2d\eta}{\alpha_1(x)m''_x}\right)$ holds true. A first order derivation of p'_x on variable $\mathfrak{f}$ then yields $\mathfrak{a}'p'_x(\mathfrak{f}') \leq \mathfrak{a}p'_x(\mathfrak{f}) + \mathfrak{a}p''_x(\mathfrak{f})\frac{2d\eta}{\alpha_1(x)m''_x}$. Now, $\frac{\mathfrak{a}p''_x(\mathfrak{f})}{\alpha_1(x)m''_x} \geq 1$ and consequently $\mathfrak{a}p''_x(\mathfrak{f})\frac{2d\eta}{\alpha_1(x)m''_x} \leq 2d\eta < d\eta$. Finally, it follows $\mathfrak{a}'p'_x(\mathfrak{f}') < \mathfrak{a}p'_x(\mathfrak{f}) + d\eta$. Two cases appear. Before going on with them, remark that $\mathfrak{f}' < \mathfrak{f}$, consequently to hypothesis $\mathfrak{f}' \leq \mathfrak{f} + \frac{2d\eta}{\alpha_1(x)m''_x}$. **First case:** $\mathfrak{f} > \varphi_1(x)$. Then, by definition of H^x_η, $\mathfrak{a} \geq \frac{\eta}{p'_x(\mathfrak{f})}$ holds true. Thus $\mathfrak{a}p'_x(\mathfrak{f}) \leq \eta$ and $\mathfrak{a}'p'_x(\mathfrak{f}') < \eta + d\eta$. Properties $\mathfrak{f}' < \mathfrak{f}$ and $\mathfrak{f} \leq \varphi_2(x)$ yields $\mathfrak{f}' < \varphi_2(x)$. Accordingly to definition of $H^x_{\eta+d\eta}$, property $(\mathfrak{a}',\mathfrak{f}') \notin H^x_{\eta+d\eta}$ holds from $\mathfrak{f}' < \varphi_2(x)$ and $\mathfrak{a}'p'_x(\mathfrak{f}') < \eta + d\eta$. **Second case:** $\mathfrak{f} = \varphi_1(x)$. Then hold $\mathfrak{f}' < \varphi_1(x)$ and consequently $(\mathfrak{a}',\mathfrak{f}') \notin H^x_{\eta+d\eta}$.

□□□

Implications (26) and (27) are now proven. Three case are then considered (refer to figure 10). **First**, if $\mathfrak{f} = \varphi_1(x)$, definition (12) yields $(\mathfrak{a},\mathfrak{f}) \in H^x_\eta \Rightarrow (\mathfrak{a},\mathfrak{f}) \in H^x_{\eta+d\eta}$, whenever $d\eta < 0$. Thus, choices $\delta\mathfrak{a} = 0$ and $\delta\mathfrak{f} = 0$ are fitting and lemma holds. **Secondly**, assume $\varphi_1(x) < \mathfrak{f} < \varphi_2(x)$. Hypothesis $\mathfrak{f}' \leq \mathfrak{f}$ then yields $\mathfrak{f}' < \varphi_2(x)$. In other word, the only two hypothesis $\mathfrak{f}' \leq \mathfrak{f}$ and $\mathfrak{a}' \geq \mathfrak{a} + \frac{2d\eta}{p'_x(\varphi_2(x))}$ are then sufficient to fulfill implication (26). Combined to (27), it signifies that $(\mathfrak{a}',\mathfrak{f}') \notin H^x_{\eta+d\eta}$ (where $\mathfrak{a}' \geq \mathfrak{a}$ and $\mathfrak{f}' \leq \mathfrak{f}$), whenever $\mathfrak{a}' \geq \mathfrak{a} + \frac{2d\eta}{p'_x(\varphi_2(x))}$ or $\mathfrak{f}' \leq \mathfrak{f} + \frac{2d\eta}{\alpha_1(x)m''_x}$. It follows that variations $\delta\mathfrak{a}$ and $\delta\mathfrak{f}$ verify $\delta\mathfrak{a} \leq \frac{2d\eta}{p'_x(\varphi_2(x))}$ and $\delta\mathfrak{f} \geq \frac{2d\eta}{\alpha_1(x)m''_x}$. Again, lemma holds. **At last**, make hypothesis $\mathfrak{f} = \varphi_2(x)$. Compared to previous case, hypothesis $\mathfrak{f}' < \varphi_2(x)$ is necessary to check implication (26). Combined to (27), it signifies that either $\delta\mathfrak{f} = 0$ with no constraint on $\delta\mathfrak{a}$, or $0 > \delta\mathfrak{f} \geq \frac{2d\eta}{\alpha_1(x)m''_x}$ with constraint $\delta\mathfrak{a} \leq \frac{2d\eta}{p'_x(\varphi_2(x))}$ on $\delta\mathfrak{a}$. Hypothesis $\delta\mathfrak{f} \neq 0$ yields also the lemma. But what happen if $\delta\mathfrak{f} = 0$? In such situation, curve

Λ^x_λ intersects both curves H^x_η and $H^x_{\eta+d\eta}$ on their upper flat part (refer to figure 10). On the one hand, suppose $\mathfrak{a} < \frac{\eta+d\eta}{p'_x(\varphi_2(x))}$. Then $\left(\frac{\eta+d\eta}{p'_x(\varphi_2(x))}, \mathfrak{f}\right) \in \Lambda^x_\lambda$ (the flat part of Λ^x_λ is an interval). Hence $\left(\frac{\eta+d\eta}{p'_x(\varphi_2(x))}, \mathfrak{f}\right) \in H^x_{\eta+d\eta} \cap \Lambda^x_\lambda$. Now, definition of H^x_η yields $\mathfrak{a} \geq \frac{\eta}{p'_x(\varphi_2(x))}$. Thus $\frac{\eta+d\eta}{p'_x(\varphi_2(x))} - \mathfrak{a} \leq \frac{d\eta}{p'_x(\varphi_2(x))} < \frac{2d\eta}{p'_x(\varphi_2(x))}$. In other word, the choices $\delta\mathfrak{a} = \frac{\eta+d\eta}{p'_x(\varphi_2(x))} - \mathfrak{a}$ and $\delta\mathfrak{f} = 0$ are fitting and yield the lemma. On the other hand, suppose $\mathfrak{a} \geq \frac{\eta+d\eta}{p'_x(\varphi_2(x))}$. Consequently, $(\mathfrak{a}, \mathfrak{f}) \in H^x_{\eta+d\eta}$. Since then $(\mathfrak{a}, \mathfrak{f}) \in H^x_{\eta+d\eta} \cap \Lambda^x_\lambda$, variations $\delta\mathfrak{a} = 0$ and $\delta\mathfrak{f} = 0$ are fitting and lemma holds.

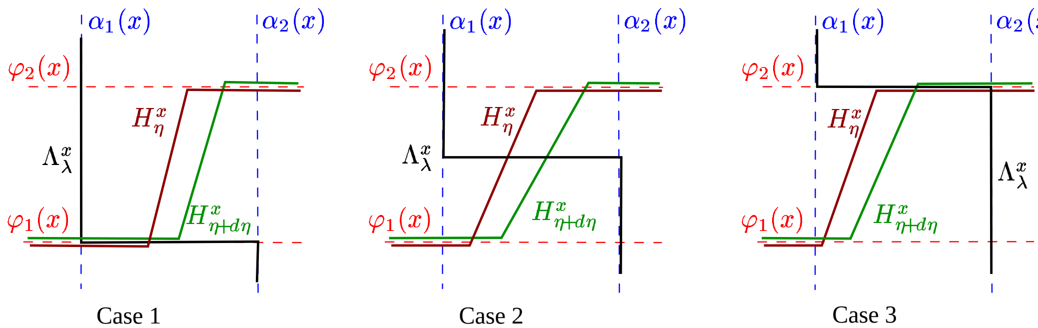


Figure 10: Proof of lemma 2

C Convexity of $w^\rho_x = -\log p^\rho_x$

When search efforts vary from φ to $\varphi + d\varphi$, the non-detection probability may be rewritten:

$$p^\rho_x(\varphi + d\varphi) = p^\rho_x(\varphi) p^\rho_x(d\varphi|\varphi) ,$$

where $p^\rho_x(d\varphi|\varphi)$ represents the elementary probability of non-detection for a new effort $d\varphi$, knowing that φ resources have already been in use. It is assumed in this paper, that $p^\rho_x(d\varphi|\varphi)$ is constant or increases with φ. The last case means that resources concentration lowers the detection power of these resources: **detection holds with waste**. On the other hand, the first case means that the detection power of the resources does not depend on their concentration: **detection holds without waste**. This hypothesis is commonly used in the literature. Now, writting

$p_x^\rho(d\varphi|\varphi) = 1 - \omega_x^\rho(\varphi)d\varphi$, the following is obtained:

$$\frac{dp_x^\rho}{p_x^\rho} = -\omega_x^\rho(\varphi)d\varphi \ .$$

It follows $\frac{d\log p_x^\rho}{d\varphi} = -\omega_x^\rho(\varphi)$. Increaseness hypothesis made on $p_x^\rho(d\varphi|\varphi)$ yields the decreseness of $\omega_x^\rho(\varphi)$. Then holds the $\boxed{\text{concavity of } w_x^\rho = -\log p_x^\rho}$.

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